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OBSERVATIONS ON THE HYPERBOLA $y^2 = 87x^2 - 6$

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Abstract: The binary quadratic equation represented by the negative pellian $y^2 = 87x^2 - 6$ is analysed for its distinct integer solutions. A few interesting relations among the solutions are given. Further, employing the solutions of the above hyperbola, we have obtained solution of other choices of hyperbolas, parabolas and special Pythagorean triangle.

Keywords: Binary quadratic, hyperbola, parabola, integral solutions, pell equation.
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I. INTRODUCTION

Diophantine equation of the form $y^2 = Dx^2 + 1$,where D is a given positive square-free integer is known as pell equation and is one of the oldest Diophantine equation that has interesting Mathematicians all over the world, since antiquity, J.L Lagrange proved that the positive pell equation $y^2 = Dx^2 + 1$ has infinitely many distinct integer solutions whereas the negative pell equation $y^2 = Dx^2 - 1$ does not always have a solution. In [1], an elementary proof of a criterium for the solvability of the pell equation $x^2 - Dy^2 = -1$ where D is any positive non-square integer has been presented. For examples the equations $y^2 = 3x^2 - 1$, $y^2 = 7x^2 - 4$ have no integer solution whereas $y^2 = 54x^2 - 1$, $y^2 = 202x^2 - 1$ have integer solutions. In this context, one may refer [2-17]. More specifically, one may refer "The on-line encyclopedia of integer sequences" (A031396, A130226, A031398) for values of D for which the negative pell equation $y^2 = Dx^2 - 1$ is solvable or not.

In this communication, the negative pell equation given by $y^2 = 87x^2 - 6$ is considered and infinitely many integer solutions are obtained. A few interesting relations among the solutions are presented.

II.METHOD OF ANALYSIS:

The negative pell equation representing hyperbola under consideration is

$$y^2 = 87x^2 - 6 \qquad (1)$$

whose smallest positive integer is $x_0 = 1, y_0 = 9$

To obtain the other solution is of (1), consider the pell equation

$$y^2 = 87x^2 + 1 \qquad (2)$$

whose initial solution is $\tilde{x}_0 = 3, \tilde{y}_0 = 28$

The general solution $(\tilde{x}_n, \tilde{y}_n)$ of (2) is given by

$$\tilde{x}_n = \frac{1}{2\sqrt{87}} g_n \,,$$

$$\tilde{y}_n = \frac{1}{2} f_n$$

Where,

$$f_n = \left(28+3\sqrt{87}\right)^{n+1} + \left(28-3\sqrt{87}\right)^{n+1}$$

$$g_n = \left(28+3\sqrt{87}\right)^{n+1} - \left(28-3\sqrt{87}\right)^{n+1}$$

Applying the Lemma of Brahamagupta between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the other solutions of (1) are given by

$$x_{n+1} = \frac{1}{2} f_n + \frac{9}{8\sqrt{87}} g_n \,,$$

$$y_{n+1} = \frac{9}{2} f_n + \frac{\sqrt{87}}{2} g_n \quad , \text{n=0,1,2…….}$$

The recurrence relations satisfied by the solutions x and y are given by

$$x_{n+3} - 56x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 56y_{n+2} + y_{n+1} = 0$$

Some numerical examples of x and y satisfying (1) are given in the Table 1 below

TABLE 1: Examples

n	x_n	y_n
0	1	9
1	55	513
2	3079	28719
3	172369	1607751
4	9649585	90005337
5	540204391	5038691121

From the above, we observe some interesting relations among the solutions which are presented below.

1. Both the values of x_n and y_n are odd.
2. Each of following expressions is a nasty number

❖ $6\left[57x_{2n+2} - x_{2n+3} + 2\right]$

❖ $\dfrac{3\left[3191x_{2n+2} - x_{2n+4} + 112\right]}{28}$

❖ $\dfrac{3\left[1595x_{2n+2} - 3y_{2n+3} + 56\right]}{14}$

❖ $\dfrac{6\left[89291x_{2n+2} - 3y_{2n+4} + 3134\right]}{1567}$

❖ $6\left[3191x_{2n+3} - 57x_{2n+4} + 2\right]$

- $\frac{3[29x_{2n+3}-1711y_{2n+2}+56]}{14}$
- $6[1595x_{2n+3}-1711y_{2n+3}+2]$
- $\frac{3[89291x_{2n+3}-1711y_{2n+4}+56]}{14}$
- $\frac{6[29x_{2n+4}-9573y_{2n+2}+3134]}{1567}$
- $\frac{3[1595x_{2n+4}-9573y_{2n+3}+56]}{14}$
- $6[89291x_{2n+4}-9573y_{2n+4}+2]$
- $\frac{2[y_{2n+3}-55y_{2n+2}+18]}{3}$
- $\frac{[y_{2n+4}-3079y_{2n+2}+1008]}{84}$
- $\frac{2[55y_{2n+4}-3079y_{2n+3}+18]}{3}$

3. Each of the following expressions is a cubical integer

- $57x_{3n+3}-x_{3n+4}+3[57x_{n+1}-x_{n+2}]$
- $3136[3191x_{3n+3}-x_{3n+5}+3(3191x_{n+1}-x_{n+3})]$
- $784[1595x_{3n+3}-3y_{3n+4}+3(1595x_{n+1}-3y_{n+2})]$
- $2455489[89291x_{2n+2}-3y_{2n+4}+3(89291x_{n+1}-3y_{n+3})]$
- $3191x_{2n+3}-57x_{2n+4}+3(3191x_{n+2}-57x_{n+3})$
- $784[29x_{2n+3}-1711y_{2n+2}+3(29x_{n+2}-1711y_{n+1})]$
- $1595x_{2n+3}-1711y_{2n+3}+3(1595x_{n+2}-1711y_{n+2})$
- $784[89291x_{2n+3}-1711y_{2n+4}+3(89291x_{n+2}-1711y_{n+3})]$
- $2455489[29x_{2n+4}-9573y_{2n+2}+3(29x_{n+3}-9573y_{n+1})]$
- $784[1595x_{2n+4}-9573y_{2n+3}+3(1595x_{n+3}-9573y_{n+2})]$
- $89291x_{2n+4}-9573y_{2n+4}+3(89291x_{n+3}-9573y_{n+3})$
- $81[y_{2n+3}-55y_{2n+2}+3(y_{n+2}-55y_{n+1})]$
- $254016[y_{2n+4}-3079y_{2n+2}+3(y_{n+3}-3079y_{n+1})]$
- $81[55y_{2n+4}-3079y_{2n+3}+3(55y_{n+3}-3079y_{n+2})]$

4. Relations among the solutions

- $x_{n+3}=56x_{n+2}-x_{n+1}$
- $3y_{n+1}=x_{n+2}-28x_{n+1}$
- $3y_{n+2}=28x_{n+2}-x_{n+1}$
- $3y_{n+3}=1567x_{n+2}-28x_{n+1}$
- $168y_{n+1}=x_{n+3}-1567x_{n+1}$
- $6y_{n+2}=x_{n+3}-x_{n+1}$
- $168y_{n+3}=1567x_{n+3}-x_{n+1}$
- $28y_{n+1}=y_{n+2}-2611x_{n+1}$

- $28y_{n+3} = 261x_{n+1} + 1567y_{n+2}$
- $1567y_{n+1} = y_{n+3} - 14616x_{n+1}$
- $3y_{n+1} = 28x_{n+3} - 1567x_{n+2}$
- $3y_{n+2} = x_{n+3} - 28x_{n+2}$
- $3y_{n+3} = 28x_{n+3} - x_{n+2}$
- $28y_{n+2} = 261x_{n+2} + y_{n+1}$
- $y_{n+3} = 522x_{n+2} + y_{n+1}$
- $y_{n+3} = 261x_{n+2} + 28y_{n+2}$
- $1567y_{n+2} = 261x_{n+3} + 28y_{n+1}$
- $1567y_{n+3} = 14616x_{n+3} + y_{n+1}$
- $28y_{n+3} = 261x_{n+3} + y_{n+2}$
- $y_{n+3} = 56y_{n+2} - y_{n+1}$

III.REMARKABLE OBSERVATIONS:

I. Employing linear combination among the solutions of (1), one may generate integer solutions for the other choices of hyperbolas which are presented in the Table 2 below

TABLE 2: Hyperbolas

S.No	(X,Y)	HYPERBOLA
1	$(x_{n+2} - 55x_{n+1}, 57x_{n+1} - x_{n+2})$	$27Y^2 - 29X^2 = 108$
2	$(x_{n+3} - 3079x_{n+1}, 3191x_{n+1} - x_{n+3})$	$27Y^2 - 29X^2 = 338688$
3	$(y_{n+2} - 513x_{n+1}, 1595x_{n+1} - 3y_{n+2})$	$3Y^2 - 29X^2 = 9408$
4	$(y_{n+3} - 28719x_{n+1}, 89291x_{n+1} - 3y_{n+3})$	$3Y^2 - 29X^2 = 2946586$
5	$(55x_{n+3} - 3079x_{n+2}, 3191x_{n+2} - 57x_{n+3})$	$27Y^2 - 29X^2 = 108$
6	$(55y_{n+1} - 9x_{n+2}, 29x_{n+2} - 171y_{n+1})$	$3Y^2 - 29X^2 = 9408$
7	$(55y_{n+2} - 513x_{n+2}, 1595x_{n+2} - 171y_{n+2})$	$3Y^2 - 29X^2 = 12$
8	$(55y_{n+3} - 28719x_{n+2}, 89291x_{n+2} - 171y_{n+3})$	$3Y^2 - 29X^2 = 9408$
9	$(3079y_{n+1} - 9x_{n+3}, 29x_{n+3} - 9573y_{n+1})$	$3Y^2 - 29X^2 = 2946586$
10	$(3079y_{n+2} - 513x_{n+3}, 1595x_{n+3} - 9573y_{n+2})$	$3Y^2 - 29X^2 = 9408$
11	$(3079y_{n+3} - 28719x_{n+3}, 89291x_{n+3} - 9573y_{n+3})$	$3Y^2 - 29X^2 = 12$
12	$(57y_{n+1} - y_{n+2}, y_{n+2} - 55y_{n+1})$	$29Y^2 - 27X^2 = 9396$
13	$(3191y_{n+1} - y_{n+3}, y_{n+3} - 3079y_{n+1})$	$1421Y^2 - 1323X^2 = 144382694$
14	$(3191y_{n+2} - 57y_{n+3}, 55y_{n+3} - 3079y_{n+2})$	$29Y^2 - 27X^2 = 9396$

Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in the Table 3 below

TABLE 3: Parabolas

S.No.	(X,Y)	PARABOLA
1	$(x_{n+2} - 55x_{n+1}, 57x_{2n+2} - x_{2n+3} + 2)$	$27Y - 29X^2 = 108$
2	$(x_{n+3} - 3079x_{n+1}, 3191x_{2n+2} - x_{2n+4} + 112)$	$1512Y - 29X^2 = 338688$

3	$(y_{n+2} - 513x_{n+1}, 1595x_{2n+2} - 3y_{2n+3} + 56)$	$84Y - 29X^2 = 9408$
4	$(y_{n+3} - 28719x_{n+1}, 89291x_{2n+2} - 3y_{2n+4} + 3134)$	$4701Y - 29X^2 = 29465868$
5	$(55x_{n+3} - 3079x_{n+2}, 3191x_{2n+3} - 57x_{2n+4} + 2)$	$27Y - 29X^2 = 108$
6	$(55y_{n+1} - 9x_{n+2}, 29x_{2n+3} - 171y_{2n+2} + 56)$	$84Y - 29X^2 = 9408$
7	$(55y_{n+2} - 513x_{n+2}, 1595x_{2n+3} - 171y_{2n+3} + 2)$	$3Y - 29X^2 = 12$
8	$(55y_{n+3} - 28719x_{n+2}, 89291x_{2n+3} - 171y_{2n+4} + 56)$	$84Y - 29X^2 = 9408$
9	$(3079y_{n+1} - 9x_{n+3}, 29x_{2n+4} - 9573y_{2n+2} + 3134)$	$4701Y - 29X^2 = 29465868$
10	$(3079y_{n+2} - 513x_{n+3}, 1595x_{2n+4} - 9573y_{2n+3} + 56)$	$84Y - 29X^2 = 9408$
11	$(3079y_{n+3} - 28719x_{n+3}, 89291x_{n+3} - 9573y_{2n+4} + 2)$	$3Y - 29X^2 = 12$
12	$(57y_{n+1} - y_{n+2}, y_{2n+3} - 55y_{2n+2} + 18)$	$29Y - 3X^2 = 1044$
13	$(3191y_{n+1} - y_{n+3}, y_{2n+4} - 3079y_{2n+2} + 1008)$	$1624Y - 3X^2 = 3273984$
14	$(3191y_{n+2} - 57y_{n+3}, 55y_{2n+4} - 3079y_{2n+3} + 118)$	$29Y - 3X^2 = 1044$

Consider $p = x_{n+1} + y_{n+1}, q = x_{n+1}$, observe that p>q>0. Treat p,q as the generators of the Pythagorean triangle $T(\alpha, \beta, \gamma)$,

$$\alpha = 2pq, \ \beta = p^2 - q^2, \ \gamma = p^2 + q^2$$

Then the following interesting relations are observed.

a) $2\alpha - 87\beta + 85\gamma = 12$

b) $89\alpha - 2\gamma - 12 = \frac{348A}{P}$

c) $\frac{2A}{P} = x_{n+1} y_{n+1}$

IV.CONCLUSION

In this paper, we have presented infinitely many solutions for the hyperbola represented by the negative pell equation $y^2 = 87x^2 - 6$. As the binary quadratic diophantine equations are rich in variety, one may search for the other choices of negative pell equations and determine their integer solutions along with suitable properties.

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