A New Integer Sequence

M.A. Gopalan¹, S. Vidhyalaksmi², J. Shanthi³

¹Professor, Department of Mathematics, Shrimati Indira Gandhi College, Tamil nadu, India,
²Professor, Department of Mathematics, Shrimati Indira Gandhi College, Tamil nadu, India,
³Asst. Professor, Department of Mathematics, Shrimati Indira Gandhi College, Tamil nadu, India,

Abstract—In this paper, a new integer sequence is developed by defining the recurrence relation

\[ J_{n+2} = J_{n+1} + (k^2 + k)J_n \]

with the initial conditions \( J_0 = 0, J_1 = 1 \). Various interesting relations among these numbers are exhibited. Also, Diophantine quadruples with property \( D(k^{2n}) \) are constructed. Some numerical examples are given.

Keywords: Integer sequence, Binet’s formula

I. INTRODUCTION

It is well known that the Fibonacci sequence is famous for its wonderful and amazing properties. Fibonacci composed a number text in which he did important work in number theory and the solution of algebraic equations. The equation of rabbit problem posed by Fibonacci is known as the first mathematical model for population growth. From the statement of rabbit problem, the famous Fibonacci numbers can be derived. This sequence of Fibonacci numbers is extremely fruitful and appears in different areas in mathematics and science.

The Fibonacci sequence, Lucas sequence, Pell sequence, Pell-Lucas sequence, Jacobsthal sequence and Jacobsthal-Lucas sequence are most prominent examples of recursive sequences.

The Fibonacci sequence [3] is defined by the recurrence relation \( F_k = F_{k-1} + F_{k-2}, k \geq 2 \) with \( F_0 = 0, F_1 = 1 \). The Lucas sequence [3] is defined by the recurrence relation \( L_k + L_{k-1} + L_{k-2}, k \geq 2 \) with \( L_0 = 2, L_1 = 1 \).

The second order recurrence sequence has been generalized in two ways mainly, first by preserving the initial conditions and second by preserving the recurrence relation. In this context, one may refer [9].

D. Kalman and R. Mena[2] generalized the Fibonacci sequence by \( F_n = aF_{n-1} + bF_{n-2}, n \geq 2 \) with \( F_0 = 0, F_1 = 1 \).

A. F. Horadam[1] defined generalized the Fibonacci sequence \( \{ H_n \} \) by \( H_n = H_{n-1} + H_{n-2}, n \geq 3 \) with \( H_1 = pH_2 = p + q \), where \( p \) and \( q \) are arbitrary integers.

B. Singh, O. Sikhwal and S. Bhatnagar [12], defined Fibonacci like sequence by recurrence relation \( S_k = S_{k-1} + S_{k-2}, k \geq 3 \) with \( S_0 = 2, S_1 = 2 \). The associated initial conditions \( S_0 \) and \( S_1 \) are the sum of the Fibonacci and Lucas sequence respectively. i.e, \( S_0 = F_0 + L_0 \) and \( S_1 = F_1 + L_1 \).

L.R. Natividad [11], Deriving a formula in solving Fibonacci like sequence. He found missing terms in Fibonacci like sequence and solved by standard formula.
V.K. Gupta, V.Y. Panwar and O. Sikhwal [7], defined generalized Fibonacci sequences and derived its identities connection formulae and other results. V.K. Gupta, V.Y. Panwar and N.Gupta [6], stated and derived identities for Fibonacci like sequence. Also, described and derived connection formulae and negation formulae for Fibonacci like sequence. B.Singh, V.K.Gupta and V.Y.Panwar [4], present many combination of higher powers of Fibonacci like sequence.

The k-Fibonacci numbers defined by Falco ‘n’ Plaza.A [5], depending only an one integer parameter k as follows, for any positive real number k, the Fibonacci sequence is defined recurrently by

\[ F_{nk} = kF_{k,n-1} + F_{k,n-2}, \quad n \geq 2 \]  \quad \text{with} \quad F_{k,0} = 0, F_{k,1} = 1.

In [10], A.D. Godase and M.B. Dhakne have presented some properties of k-Fibonacci and k-Lucas numbers by using matrices.

In [8], Yashwant, K.Panwar, G.P. Rathore and Richa Chawla have established some interesting properties of k-Fibonacci like numbers.

The above results motivated us to search for new integer sequences with suitable initial conditions.

In this communication, a new integer sequence \( \{J_n\} \) is developed by defining the recurrence relation

\[ J_{n+2} = J_{n+1} + (k^2 + k) J_n \]  \quad \text{with the initial condition} \quad J_0 = 0, J_1 = 1.

Various interesting relations among these numbers are exhibited. Also, Diophantine quadruples with property \( D(k^{2n}) \) are constructed. Some numerical examples are given.

\section*{II. \hspace{2em} METHOD OF ANALYSIS}

Consider a sequence \( \{J_n\} \) defined by

\[ J_{n+2} = J_{n+1} + (k^2 + k) J_n \]  \quad \text{(1)}

with the initial condition \( J_0 = 0, J_1 = 1 \).

The auxiliary equation associated with the recurrence relation (1) is given by

\[ m^2 - m - (k^2 + k) = 0 \]

whose roots are

\[ \alpha + \beta = 1, \alpha\beta = -(k^2 + k) \]

Thus the general solution of (1) is \( J_n = A\alpha^n + B\beta^n \)

From the initial conditions we infer that

\[ A + B = 0, A\alpha + B\beta = 1 \]

Solving for A and B, we get

\[ A = -\frac{1}{\alpha - \beta}, B = \left( \frac{1}{\alpha - \beta} \right) \]

Thus a notable sequence \( \{J_n\} \) whose terms are given below is obtained.

\[ J_n = \frac{(k + 1)^n - (-k)^n}{2k + 1} \]  \quad \text{(2)}

The first ten numbers are given below:
\[ J_n = 0 \]
\[ J_1 = 1 \]
\[ J_2 = 1 \]
\[ J_3 = 1 + k + k^2 \]
\[ J_4 = 1 + 2k + 2k^2 \]
\[ J_5 = 1 + 3k + 4k^2 + 2k^3 + k^4 \]
\[ J_6 = 1 + 4k + 7k^2 + 6k^3 + 3k^4 \]
\[ J_7 = 1 + 5k + 11k^2 + 13k^3 + 9k^4 + 3k^5 + k^6 \]
\[ J_8 = 1 + 6k + 16k^2 + 24k^3 + 22k^4 + 12k^5 + 4k^6 \]
\[ J_9 = 1 + 7k + 22k^2 + 40k^3 + 46k^4 + 34k^5 + 16k^6 + 4k^7 + k^8 \]
\[ J_{10} = 1 + 8k + 29k^2 + 62k^3 + 86k^4 + 80k^5 + 50k^6 + 20k^7 + 5k^8 \]

The new sequence \( \{J_n\} \) is found to satisfy the following relations:

Relations:

1. \[ 3[J_{2n+1} + k(k+1)J_{2n-1} + (2k+1)J_n] \] is a Nasty Number.

Proof:
\[
J_{n+1} + k(k+1)J_{n-1} = \frac{1}{(2k+1)}[(k+1)(k+1)^n - (-k)(-k)^n + k(k+1)^n + (k+1)(-k)^n]
\]
\[
= (k+1)^n + (-k)^n
\]
\[
= (2k+1)J_n + 2(-k)^n \quad \text{from (2)}
\]
\[
= 2(k+1)^n + (2k+1)J_n
\]
Replacing \( n \) by \( 2n \) in the above equation, we get
\[
J_{2n+1} + k(k+1)J_{2n-1} = 2(k+1)^{2n} - (2k+1)J_{2n}
\]
\[
\therefore 3[J_{2n+1} + k(k+1)J_{2n-1} + (2k+1)J_n] = 6(k+1)^{2n}
\]
Hence \( 3[J_{2n+1} + k(k+1)J_{2n-1} + (2k+1)J_n] \) is a Nasty Number.

2. \[ J_{m+n} + (2k+1)J_mJ_n = (k+1)^m J_n + (k+1)^n J_m \]

Proof:
\[
J_{m+n} + (2k+1)J_mJ_n = \frac{1}{(2k+1)}[2(k+1)^{m+n} - (k+1)^m (-k)^n - (k+1)^n (-k)^m]
\]
\[
= \frac{1}{(2k+1)}[(k+1)^m((k+1)^n - (-k)^n) + (k+1)^n((k+1)^m - (-k)^m)]
\]
\[
= (k+1)^m J_n + (k+1)^n J_m
\]
\[
\therefore J_{m+n} + (2k+1)J_mJ_n = (k+1)^m J_n + (k+1)^n J_m.
\]
3. \( J_{m+n} - (2k+1)J_mJ_n = (-k)^mJ_n + (-k)^nJ_m \)

**Proof:**

\[
J_{m+n} - (2k+1)J_mJ_n = \frac{1}{(2k+1)}[(k+1)^m(-k)^n + (k+1)^n(-k)^m - 2(-k)^{m+n}]
\]

\[
= (-k)^mJ_n + (-k)^nJ_m
\]

\[
\therefore J_{m+n} - (2k+1)J_mJ_n = (-k)^mJ_n + (-k)^nJ_m.
\]

4. \( J_{n+1}^2 - k^2(k+1)^2J_n^2 = J_{2n} \)

**Proof:**

\[
J_{n+1}^2 - k^2(k+1)^2J_n^2 = \frac{1}{(2k+1)^2}[(k+1)^{2n+2} + (-k)^{2n+2} - 2(k+1)^{n+1}(k)^{n+1}]
\]

\[
= \frac{1}{(2k+1)^2}[k^2(k+1)^{2n} + (k+1)^2(-k)^{2n} - 2(k+1)^{n+1}(-k)^{n+1}]
\]

\[
= \frac{1}{(2k+1)^2}[(k+1)^{2n}(2k+1) - (-k)^{2n}(2k+1)]
\]

\[
= J_{2n}
\]

\[
\therefore J_{n+1}^2 - k^2(k+1)^2J_n^2 = J_{2n}.
\]

5. \( J_{n+1}^2 - k^2(k+1)^2J_n^2 - 2(k+1)^2 = (2k^2 + 2k + 1)[J_{2n+1} + k(k+1)J_{2n-1}] - 4(k+1)^{n+1}(-k)^{n+1} \)

**Proof:**

\[
J_{n+1}^2 - k^2(k+1)^2J_n^2 - 2(k+1)^2 = (k+1)^{2n+2} + (-k)^{2n+2} - 2(k+1)^{n+1}(k)^{n+1}
\]

\[
+ [k^2(k+1)^{2n} + (k+1)^2(-k)^{2n} - 2(k+1)^{n+1}(-k)^{n+1}]
\]

\[
= (2k^2 + 2k + 1)[J_{2n+1} + k(k+1)J_{2n-1}] - 4(k+1)^{n+1}(-k)^{n+1}
\]

Hence,

\[
J_{n+1}^2 - k^2(k+1)^2J_n^2 - 2(k+1)^2 = (2k^2 + 2k + 1)[J_{2n+1} + k(k+1)J_{2n-1}] - 4(k+1)^{n+1}(-k)^{n+1}.
\]

6. \( J_mJ_{n+1} - J_nJ_{m+1} = (-k)^nJ_m - (-k)^mJ_n \)

**Proof:**

\[
J_mJ_{n+1} - J_nJ_{m+1} = \frac{1}{(2k+1)^2}[(k+1)^n(-k)^n(-k - k - 1) + (k+1)^m(-k)^m(2k + 1)]
\]

\[
= \frac{1}{(2k+1)^2}[(k+1)^n(-k)^n - (k + 1)^n(-k)^n]
\]

\[
= \frac{1}{(2k+1)^2}[-(-k)^n(k+1)^n + (-k)^{n+1}(k)^{n+1}]
\]

\[
= (-k)^nJ_m - (-k)^mJ_n
\]
Hence, \( J_{n+1} J_{m+1} - J_n J_{m+1} = (-k)^n J_m - (-k)^n J_n \).

7. \((2k+1)^2 J_{n+1} J_{n-1} = J_{2n+1} + k(k+1) J_{2n-1} (k+1)^{n-1} (-k)^{n-1} (2k^2 + 2k+1)\)

Proof:
\[
(2k+1)^2 J_{n+1} J_{n-1} = \left[ (k+1)^{n+1} - (-k)^{n+1} \right] \left[ (k+1)^{n-1} - (-k)^{n-1} \right]
= (k+1)^{2n} - (k+1)^{n+1} (-k)^{n-1} - (k+1)^{n-1} (-k)^{n+1} + (-k)^{2n}
= (k+1)^{2n} + (-k)^{2n} + (k+1)^n (-k)^n \left[ \frac{k+1}{k} + \frac{k}{k+1} \right]
= J_{2n+1} + k(k+1) J_{2n-1} (k+1)^{n-1} (-k)^{n-1} (2k^2 + 2k+1)
\]

Hence, \((2k+1)^2 J_{n+1} J_{n-1} = J_{2n+1} + k(k+1) J_{2n-1} (k+1)^{n-1} (-k)^{n-1} (2k^2 + 2k+1)\)

8. \((2k+1) J_{n+1}^2 + 2(-k)^{n+1} J_{n+1} = J_{2n+2}\)

Proof:
\[
(2k+1) J_{n+1}^2 = (2k+1) J_{2n+2} + 2(-k)^{2n+2} - 2(k+1)^{n+1} (-k)^{n+1}
= (2k+1) J_{2n+2} - 2(-k)^{n+1} \left[ (k+1)^{n+1} - (-k)^{n+1} \right]
= (2k+1) J_{2n+2} - 2(-k)^{n+1} (2k+1) J_{n+1}
\]

\[\therefore \ (2k+1) J_{n+1}^2 + 2(-k)^{n+1} J_{n+1} = (2k+1) J_{2n+2}\]

Hence, \((2k+1) J_{n+1}^2 + 2(-k)^{n+1} J_{n+1} = J_{2n+2}\).

9. \((2k+1) J_{n+r-1} J_{n-r} = J_{2n} - (-k)^{n-r} J_{n+r} - (-k)^{n+r} J_{n-r}\)

Proof:
\[
(2k+1) J_{n+r-1} J_{n-r-1} = (k+1)^{2n} + (-k)^{2n} - (k+1)^{n+r} (-k)^{n-r} - (k+1)^{n-r} (-k)^{n+r}
= (k+1)^{2n} + (-k)^{2n} - (k+1)^{n+r} \left[ (k+1)^{n-r} - (-k)^{n+r} \right] - (-k)^{2n}
- (-k)^{n+r} \left[ (k+1)^{n-r} - (-k)^{n-r} \right] - (-k)^{2n}
= (k+1)^{2n} + (-k)^{2n} - (k+1)^{n+r} J_{n+r} - (-k)^{n+r} (2k+1) J_{n-r}
\]

Hence, \((2k+1) J_{n+r-1} J_{n-r-1} = J_{2n} - (-k)^{n-r} J_{n+r} - (-k)^{n+r} J_{n-r}\).

10. \( S_n = \sum_{i=0}^{n-1} J_i = \frac{J_{n+1} - 1}{k(k+1)} \)

Proof:
\[
\text{Let } S_n = \sum_{i=0}^{n-1} J_i = \frac{1}{(2k+1)} \sum_{i=0}^{n-1} \left[ (k+1)^i - (-k)^i \right]
\]
\[
J_{n+1} - 1 \over k(k+1)
\]

Hence, \( S_n = \sum_{i=0}^{n-1} J_i = J_{n+1} - 1 \over k(k+1) \).

11. \( 4(J_{3n} - J_n J_{2n+1} - k(k+1)J_n J_{2n-1}) = J_{n+1}^2 + k^2 (k+1)^2 J_{n-1}^2 + 2k(k+1)J_{n+1} J_{n-1} - (2k+1)^2 J_n^2 \)

**Proof:**
Replacing \( n \) by \( 3n \) in (2), we have

\[
J_{3n} = \frac{1}{(2k+1)} [(k+1)^{3n} - (-k)^{3n}]
\]

\[
= \frac{1}{(2k+1)} [(k+1)^n - (-k)^n (k+1)^{2n} + (-k)^2n + (k+1)^n (-k)^n]
\]

\[
= J_n + k(k+1) J_{2n-1} + \bigg\{ J_{n+1} + k(k+1) J_{n-1} + (2k+1) J_n \bigg\} \over 2
\]

Hence,

\[
4(J_{3n} - J_n J_{2n+1} - k(k+1)J_n J_{2n-1}) = J_{n+1}^2 + k^2 (k+1)^2 J_{n-1}^2 + 2k(k+1)J_{n+1} J_{n-1} - (2k+1)^2 J_n^2.
\]

12. \( J_{4n} = J_{2n} (J_{2n+1} + k(k+1) J_{2n-1}) \).

**Proof:**
Replacing \( n \) by \( 4n \) in (2), we have

\[
J_{4n} = \frac{1}{(2k+1)} [(k+1)^{4n} - (-k)^{4n}]
\]

\[
= \frac{1}{(2k+1)} [(k+1)^{2n} - (-k)^{2n} (k+1)^{2n} + (-k)^{2n}]
\]

Hence, \( J_{4n} = J_{2n} (J_{2n+1} + k(k+1) J_{2n-1}) \).

13. \( J_{4n} = J_{2n} \left[ J_{n+1} + k(k+1) J_{n-1} \right] \left[ J_{2n+1} + k(k+1) J_{2n-1} \right] \)

**Proof:**
Replacing \( n \) by \( 4n \) in (2), we have

\[
J_{4n} = \frac{1}{(2k+1)} [(k+1)^{4n} - (-k)^{4n}]
\]
\[
\frac{1}{2(2k+1)}[(k+1)^n - (-k)^n] \]

Hence, \( J_{n+1} = J_{n+1}^2 + k(k+1)J_{n+1} + k(k+1)J_{n-1} + k(k+1)J_{2n-1} \).

14. \( J_{m+n} + (\alpha - \beta)J_mJ_n = \alpha^mJ_n + \alpha^nJ_m \).

**Proof:**

From (2), we have

\[
J_m = \frac{1}{2(2k+1)}[(k+1)^m - (-k)^m]
\]

\[
J_{m+n} = \frac{1}{2(2k+1)}[(k+1)^{m+n} - (-k)^{m+n}]
\]

\[
\therefore J_{m+n} + (\alpha - \beta)J_mJ_n = \frac{1}{\alpha - \beta}[2\alpha^{m+n} - \alpha^m\beta^n - \alpha^n\beta^m]
\]

\[
= \frac{1}{\alpha - \beta}[\alpha^m(\alpha^n - \beta^n) + \alpha^n(\alpha^m - \beta^m)]
\]

Hence, \( J_{m+n} + (\alpha - \beta)J_mJ_n = \alpha^mJ_n + \alpha^nJ_m \).

**Construction of Diophantine quadruple with property** \( D(k^{2n}) \):

Let \( a = (2k+1)J_n = (k+1)^n - (-k)^n \), \( b = (J_{n+1} + k(k+1)J_{n-1} = (k+1)^n + (-k)^n \)

it is noted that \( ab + k^{2n} = [(k+1)^n]^2 = p^2 \) (say).

Therefore the pair \( (a, b) \) is Diophantine -2-tuple with property \( D(k^{2n}) \).

Let \( c = a + b + 2p = 4(k+1)^n \)

it is noted that the triple \( (a, b, c) \) is Diophantine-3-tuple with property \( D(k^{2n}) \).

By Euler’s formula the fourth tuple ‘d’ is represented by

\[
d = a + b + c + \frac{2}{k^{2n}}(abc + pqr)
\]

\[
d = 6(1)^n + \frac{2}{k^{2n}}[8(k+1)^1 - 5(-k)^2(k+1)^1]
\]

After performing calculation it is seen that \( (a, b, c, d) \) is Diophantine quadruple with property \( D(k^{2n}) \).

It is worth mentioning that the value of ‘d’ is represented by a rational number for particular values of \( k \) and \( n \). A few examples are given in a tabular form below.

**Table : Examples**

<table>
<thead>
<tr>
<th>n</th>
<th>k</th>
<th>Quadruple (a, b, c, d)</th>
<th>Property ( D(k^{2n}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>(3, 1, 8, 120)</td>
<td>D(1)</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>(3, 5, 16, 1008)</td>
<td>D(1)</td>
</tr>
</tbody>
</table>
### III. CONCLUSION

In this paper, we develop a new integer sequence using the recurrence relation

$$J_{n+2} = J_{n+1} + (k^2 + k)J_n$$

with the initial condition $J_0 = 0, J_1 = 1$, and provided a few interesting relations among its members. To conclude, one may attempt to obtain a new integer sequences employing different choices for the recurrence relations with suitable initial conditions.

### REFERENCES