OPEN SETS IN BITOPOLOGICAL SPACES

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Abstract — In this paper, we introduce $\tau_1 \tau_2 - Q^*$ open sets in bitopological spaces and study its properties.

Keywords — $\tau_1 \tau_2 - Q^*$ open sets.

I. INTRODUCTION

A triple $(X, \tau_1, \tau_2)$ where $X$ is a non-empty set and $\tau_1, \tau_2$ are topologies on $X$ is called a bitopological space and Kelly initiated the study of such spaces. Maheswari and Prasad [11] introduced semi open sets in bitopological spaces in 1977.

Closed sets are fundamental objects in a topological space. For example, one can define the topology on a set by using either the axioms for the closed sets or the Kuratowski closure axioms. In 1971, Levine [10] introduced the concept of generalized closed sets in topological spaces. Also he introduced the notion of semi open sets in topological spaces. Bhattacharyya and Lahiri [3] introduced a class of sets called semi generalized closed sets by means of semi open sets of Levine and obtained various topological properties.

In 1985, Fukutake [7] introduced the concepts of $g$-closed sets in bitopological spaces and after that several authors turned their attention towards generalizations of various concepts of topology by considering bitopological spaces.

In 2004 [19], Sheik John M and Sundaram P introduced $g^*$ closed sets in bitopological spaces. The notion of $Q^*$-closed sets in a topological space was introduced by Murugalingam and Lalitha [12] in 2010.

Recently, P. Padma and S. Udayakumar [15] introduced the concept of $(\tau_1, \tau_2)^* - Q^*$ closed sets in bitopological spaces.

In the present paper, we introduced $\tau_1 \tau_2 - Q^*$ open sets in bitopological spaces and studied some of their bitopological properties. Also some relations are established with known generalized closed sets.

II. PRELIMINARIES

Throughout this paper $X$ and $Y$ always represent nonempty bitopological spaces $(X, \tau_1, \tau_2)$ and $(Y, \sigma_1, \sigma_2)$. For a subset $A$ of $X$, $\tau_1 - \text{cl} (A)$, $\tau_1 - Q^* \text{cl} (A)$ (resp. $\tau_1 - \text{int} (A)$), $\tau_1 - Q^* \text{int} (A)$ represents closure of $A$ and $Q^*$ closure of $A$ (resp. interior of $A$, $Q^*$-interior of $A$) with respect to the topology $\tau_1$. We shall now require the following known definitions.

Definition 2.1 - A set $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is called

a) $\tau_1 \tau_2$- semi open if there exists an $\tau_1$-open set $U$ such that $U \subseteq A \subseteq \tau_2 - \text{cl} (A)$. Equivalently, a set $A$ is $\tau_1 \tau_2$-semi open if $A \subseteq \tau_2 - \text{cl} (\tau_1 - \text{int} (A))$.

b) $\tau_1 \tau_2$- semi closed if $X - A$ is $\tau_1 \tau_2$-semi open.

c) $\tau_1 \tau_2$- generalized open ($\tau_1 \tau_2$- $g$ open) if $X - A$ is $\tau_1 \tau_2$-generalized closed.

d) $\tau_1 \tau_2$- generalized closed ($\tau_1 \tau_2$- $g$ closed) if $\tau_2 - \text{cl} (A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\tau_1$-open in $X$. 

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e) $\tau_1 \tau_2$-generalized open ( $\tau_1 \tau_2$-g open ) if $X - A$ is $\tau_1 \tau_2$-g closed.

f) $\tau_1 \tau_2$-semi generalized closed ( $\tau_1 \tau_2$-sg closed ) if $\tau_2$-scl ( $A$ ) $\subseteq U$ whenever $A \subseteq U$ and $U$ is $\tau_1$-semi open in $X$.

g) $\tau_1 \tau_2$-semi generalized open ( $\tau_1 \tau_2$-sg open ) if $X - A$ is $\tau_1 \tau_2$-sg closed.

h) $\tau_1 \tau_2$-generalized semi closed ( $\tau_1 \tau_2$-gs closed ) if $\tau_2$-cl ( $A$ ) $\subseteq U$ whenever $A \subseteq U$ and $U$ is $\tau_1$-open in $X$.

i) $\tau_1 \tau_2$-generalized semi open ( $\tau_1 \tau_2$-gs open ) if $X - A$ is $\tau_1 \tau_2$-gs closed.

j) $\tau_1 \tau_2$-regular open if $A = \tau_1 \tau_2$-int [ $\tau_2$-cl ( $A$ ) ].

k) $\tau_1 \tau_2$-regular closed if $A = \tau_1 \tau_2$-cl [ $\tau_2$-int ( $A$ ) ].

l) $\tau_1 \tau_j$-g* closed sets if $\tau_j$-cl ( $A$ ) $\subseteq U$ whenever $A \subseteq U$ and $U$ is $\tau_1$-g open in $X$.

m) $\tau_1 \tau_j$-g* open ( $\tau_1 \tau_j$-g* open ) if $X - A$ is $\tau_1 \tau_j$-g* closed.

### III. $\tau_1 \tau_j$- $Q^*$S OPEN SETS

In this section, the concepts of $\tau_1 \tau_j$- $Q^*$s open sets are introduced and their basic properties in bitopological spaces are discussed. Recall that a set $A$ is a bitopological space ( $X$, $\tau_1$, $\tau_2$) is called $\tau_1 \tau_j$- $Q^*$ closed if $\tau_1$-int ( $A$ ) = $\phi$ and $A$ is $\tau_2$-closed. The family of all $\tau_1 \tau_j$- $Q^*$s closed subsets of a bitopological space ( $X$, $\tau_1$, $\tau_2$) is denoted by $\tau_1 \tau_j$- $Q^*$s.

**Definition 3.1** - A subset $A$ of a bitopological space ( $X$, $\tau_1$, $\tau_2$) is called $\tau_1 \tau_j$- $Q^*$ closed if $\tau_1$-int ( $A$ ) = $\phi$ and $A$ is $\tau_2$-closed.

**Example 3.1** - Let $X = \{ a, b, c \}$, $\tau_1 = \{ \phi, X, \{ c \} \}$, $\tau_2 = \{ \phi, X, \{ a \}, \{ b \}, \{ a, b \} \}$. Hence $\phi$, $\{ a \}$ and $\{ b \}$ are $\tau_1 \tau_j$- $Q^*$ closed.

**Definition 3.2** - A subset $A$ of a bitopological spaces ( $X$, $\tau_1$, $\tau_2$) is called $\tau_1 \tau_j$- $Q^*$s open if $X - A$ is $\tau_1 \tau_j$- $Q^*$s closed in $X$.

**Example 3.2** - In example 3.1, $X$, $\{ b, c \}$, $\{ c, a \}$ are $\tau_1 \tau_j$- $Q^*$s open.

**Remark 3.1** - Since every $\tau_1 \tau_j$- $Q^*$s open set is $\tau_2$-open and every $\tau_2$-open set is $\tau_1 \tau_2$-g open, $\tau_1 \tau_2$-sg open, $\tau_1 \tau_2$-gs open. The converse need not be true in general. The following example supports our claim.

**Example 3.3** - In example 3.1, $\{ a, b \}$ is $\tau_1 \tau_2$-g open, $\tau_1 \tau_2$-sg open and $\tau_1 \tau_2$-gs open but not $\tau_1 \tau_1$- $Q^*$s open.

**Theorem 3.1** - A set $A$ of a bitopological space ( $X$, $\tau_1$, $\tau_2$) is $\tau_1 \tau_j$- $Q^*$s open if and only if $\tau_1$-cl ( $A$ ) = $X$ and $A$ is $\tau_2$-semi open.

**Proof : Necessity :** Suppose that $A$ is $\tau_1 \tau_j$- $Q^*$s open.

Then $A^c$ is $\tau_1 \tau_j$- $Q^*$s closed.

Therefore, $\tau_1$-int ( $A^c$ ) = $[ \tau_1$-cl ( $A$ ) ]$^c$ = $\phi$ and $A^c$ is $\tau_2$-semi closed.

Consequently, $\tau_1$-cl ( $A$ ) = $X$ and $A$ is $\tau_2$-semi open.

**Sufficiency :** Suppose that $\tau_1$-cl ( $A$ ) = $X$ and $A$ is $\tau_2$-semi open.

Then $[ \tau_1$-cl ( $A$ ) ]$^c$ = $\tau_1$-int ( $A^c$ ) = $\phi$ and $A^c$ is $\tau_2$-semi closed.

Consequently, $A^c$ is $\tau_1 \tau_j$- $Q^*$s closed.

This completes the proof.

**Corollary 3.1** - A set $A$ of a bitopological space ( $X$, $\tau_1$, $\tau_2$) is $\tau_1 \tau_j$- $Q^*$s open if and only if $A$ is $\tau_1$-dense and $\tau_2$-semi open.

**Theorem 3.2** - If $A$ and $B$ are $\tau_1 \tau_j$- $Q^*$s open sets then so is $A \cap B$.

**Proof :** Suppose that $A$ and $B$ are $\tau_1 \tau_j$- $Q^*$s open sets.

Then $A^c$ and $B^c$ are $\tau_1 \tau_j$- $Q^*$s closed sets.

Therefore, $A^c \cup B^c$ is $\tau_1 \tau_j$- $Q^*$s closed sets.

But $A^c \cup B^c = ( A \cup B )^c$.

Hence $A \cap B$ is $\tau_1 \tau_j$- $Q^*$s open.
Theorem 3.3
i) \( X \) is not \( \tau_1 \tau_j - Q^* \)’s closed.
ii) \( \phi \) is \( \tau_1 \tau_j - Q^* \)’s closed.
iii) \( X \) is \( \tau_1 \tau_j - Q^* \)’s open
iv) \( X \) is not \( \tau_1 \tau_j - Q^* \)’s open.

Remark 3.2 - It is obvious that every \( \tau_1 \tau_j - Q^* \)’s open set is \( \tau_2 \) - open, but the converse is not true in general.

Remark 3.3 - Every \( \tau_1 \tau_j - Q^* \)’s open is \( \tau_1 \tau_2 \) - semi open. But the converse need not be true. The following example supports our claim.

Example 3.4 - In example 3.1, \( \{ b , c \} \) is \( \tau_1 \tau_2 \) - semi open but not \( \tau_1 \tau_j - Q^* \)’s open.

Remark 3.4 - \( \tau_1 \tau_2 - g^* \) open sets and \( \tau_1 \tau_j - Q^* \)’s open sets are independent of each other in general.

It is proved in the following example.

Example 3.5 - \( X = \{ a , b , c \} \), \( \tau_1 = \{ \phi , X , \{ c \} , \{ a , c \} \} \), \( \tau_2 = \{ \phi , X , \{ a \} \} \). Then \( \{ a , c \} \) is \( \tau_1 \tau_2 - g^* \) open but not \( \tau_1 \tau_j - Q^* \)’s open set.

Remark 3.5 - \( \tau_1 \tau_2 - g^* \) open sets and \( \tau_1 \tau_j - Q^* \)’s open sets are independent of each other in general.

It is proved in the following example.

Example 3.6 - \( X = \{ a , b , c \} \), \( \tau_1 = \{ \phi , X , \{ c \} , \{ a , c \} \} \), \( \tau_2 = \{ \phi , X , \{ a \} \} \). Then \( \{ a \} \) is \( \tau_1 \tau_2 - g^* \) open but not \( \tau_1 \tau_j - Q^* \)’s open set.

Theorem 3.4 - If \( B \subseteq A \subseteq X \), where \( A \) is \( \tau_1 \tau_j - Q^* \)’s open and \( B \) is \( \tau_1 \tau_j - Q^* \)’s open in \( A \) then \( B \) is \( \tau_1 \tau_j - Q^* \)’s open in \( X \).

Proof: Since \( B \) is \( \tau_2 \) - open in \( A \), \( A \) is \( \tau_2 \) - open in \( X \) and \( B \) is \( \tau_2 \) - open in \( X \).

We claim that \( \tau_1 - \text{cl} (B) = X \).

Let \( U \) be any \( \tau_2 \) - semi open set.

Since \( \tau_1 - \text{cl} (B) \) is \( A \), \( (U \cap A) \cap B \neq \phi \).

Then \( (U \cap A) \cap B \neq \phi \).

Hence \( \tau_1 - \text{cl} (B) = X \).

Therefore, \( B \) is \( \tau_1 \tau_j - Q^* \)’s open in \( X \).

Theorem 3.5 - If \( A \) and \( B \) are \( \tau_2 \) - open sets with \( A \cap B = \phi \) then \( A \) and \( B \) are not \( \tau_1 \tau_j - Q^* \)’s open.

Proof: Since \( A \cap B = \phi \), the points of \( B \) cannot be limit points of \( A \).

Then \( \tau_1 - \text{cl} (A) \neq X \).

Hence \( A \) is not \( \tau_1 \tau_j - Q^* \)’s open.

Similarly, \( B \) is not \( \tau_1 \tau_j - Q^* \)’s open.

Theorem 3.6 - Let \( (X, \tau_1, \tau_2) \) be a hyper connected bitopological space. Let \( A \subseteq X \). If \( A \) is \( \tau_2 \) - open then \( A \) is \( \tau_1 \tau_j - Q^* \)’s open in \( X \).

Proof: It is enough to prove that \( A \) is \( \tau_1 \) - dense.

Suppose that \( \tau_1 - \text{cl} (A) \neq X \).

Then \( \{ \tau_1 - \text{cl} (A) \} \neq \phi \).

Consequently, \( A \cap \{ \tau_1 - \text{cl} (A) \} \neq \phi \).

This is a contradiction to the fact that \( (X, \tau_1, \tau_2) \) is a hyper connected bitopological space.

Hence \( A \) is \( \tau_1 \) - dense.

REFERENCES
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