A STUDY ON MINIMUM COVERING ENERGY OF A GRAPH

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Abstract: In this chapter we discuss some basic properties of minimum covering energy and derive an upper bound and a lower bound for $E_c(G)$. Compute the minimum covering energy of path and cycle.

Keywords: $E_c(G)$ – Minimum covering Energy, $A_c(G)$ – Minimum covering Matrix

$\sum a_{ii}^2 = |C|$, $\sum a_{ij}^2 = |E|$, $\sum(a_{ii} a_{jj}) = \left(\frac{|C|}{2}\right)$, $\sum(a_{ii} a_{jj} a_{kk}) = \left(\frac{|C|}{3}\right)$.

I. INTRODUCTION

The minimum covering energy of a graph introduced by Chandrashekar Adiga [1,5], is the motivation behind this project. In this dissertation this particular energy is studied. $P_n$ and $C_n$ [8, 10, 12] for different values of $n$ are considered and their minimum covering energies have been calculated and tabulated. For some theorem have been checked.

II. DEFINITIONS

2.1 Graph

A graph $G = (V, E)$ consists of a set of objects $V = \{v_1, v_2, v_3, \ldots\}$ called vertices and another set $E = \{e_1, e_2, e_3, \ldots\}$ whose elements are called edges, such that each edge $e_k$ are called the end vertices of $e_k$.

The most common representation of a graph is by means of a diagram, in which the vertices are represented as points and each edge as a line segment joining its end vertices.

2.2 Path graph

An open walk in which no vertex appears more than once is called a path.

2.3 Cycle graph

A circuit that does not contain any repetition of vertices except the starting vertex and the terminal vertex is called cycle.

2.4 Energy of a graph

The energy, $E(G)$, of a graph $G$ is defined as the sum of the absolute values of its eigen values.

2.5 Minimum Covering set

A subset $C$ of $V$ is called a covering set of $G$ if every edge of $G$ is incident to at least one vertex of $C$. Any covering set with minimum cardinality is called a minimum covering set.

2.6 Minimum Covering Matrix

Let $C$ be a minimum covering set of a graph $G$. The minimum covering matrix of $G$ is the $n \times n$ matrix $A_c(G) = (a_{ij})$, where
\[ a_{ij} = \begin{cases} 1, & \text{if } v_i v_j \in E \\ 1, & \text{if } i = j \text{ and } v_i \in C \\ 0, & \text{otherwise} \end{cases} \]

The characteristic polynomial of \( A_c(G) \) is denoted by \( f_n(G, \lambda) = \det(\lambda I - A_c(G)) \).

### 2.7 Minimum Covering Eigenvalues

The Minimum Covering Eigen values of the graph \( G \) are the eigenvalues of \( A_c(G) \). Since \( A_c(G) \) is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \).

### 2.8 Minimum Covering Energy

The Minimum Covering energy of \( G \) is then defined as

\[ E_c(G) = \sum_{i=1}^{n} |\lambda_i|. \]

### III. Compute the minimum covering energy of two graphs

#### 3.1 Path Graph

<table>
<thead>
<tr>
<th>Graph</th>
<th>Energy</th>
<th>Minimum covering set</th>
<th>Minimum covering Energy</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_4 )</td>
<td>4.4720</td>
<td>( C_1(P_4) = {v_1, v_3} ) ( C_2(P_4) = {v_2, v_3} )</td>
<td>4.9770</td>
</tr>
<tr>
<td>( P_5 )</td>
<td>5.4610</td>
<td>( C_1(P_5) = {v_2, v_4} )</td>
<td>4.9770</td>
</tr>
<tr>
<td>( P_6 )</td>
<td>6.9890</td>
<td>( C_1(P_6) = {v_2, v_4, v_5} ) ( C_2(P_6) = {v_1, v_3, v_5} )</td>
<td>7.5728</td>
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<tr>
<td>( P_7 )</td>
<td>8.0546</td>
<td>( C_1(P_7) = {v_2, v_4, v_6} )</td>
<td>8.6568</td>
</tr>
<tr>
<td>( P_8 )</td>
<td>9.5175</td>
<td>( C_1(P_8) = {v_2, v_3, v_5, v_7} ) ( C_1(P_8) = {v_2, v_4, v_6, v_8} )</td>
<td>10.3576</td>
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<tr>
<td>( P_9 )</td>
<td>10.6275</td>
<td>( C_1(P_9) = {v_2, v_4, v_6, v_8} )</td>
<td>11.4654</td>
</tr>
<tr>
<td>( P_{10} )</td>
<td>12.0532</td>
<td>( C_1(P_{10}) = {v_2, v_4, v_4, v_4} ) ( C_2(P_{10}) = {v_1, v_3, v_5, v_7, v_9} )</td>
<td>13.3704</td>
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</table>
### 3.2 Cycle Graph

<table>
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<th>Minimum covering set</th>
<th>Minimum covering Energy</th>
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<tr>
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<td></td>
<td>$C_2(C_4) = {v_2, v_4}$</td>
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<td>$C_1(C_6) = {v_1, v_3, v_5}$</td>
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<tr>
<td>$C_9$</td>
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<tr>
<td>$C_{10}$</td>
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<td>$C_2(P_{10}) = {v_1, v_3, v_5, v_7, v_9}$</td>
<td>13.5689</td>
</tr>
</tbody>
</table>

### IV. Some basic properties of minimum covering energy

#### 4.1 Theorem

If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A_c(G)$, then $\sum_{i=1}^{n} \lambda_i^2 = 2|E| + |C|$.

#### Proof

The sum of squares of the eigenvalues of $A_c(G)$ is just the trace of $A_c(G)^2$. Therefore,

$$
\sum_{i=1}^{n} \lambda_i^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} a_{ji} = 2 \sum_{i < j} (a_{ij})^2 + \sum_{i=1}^{n} (a_{i,i})^2 = 2|E| + |C|.
$$
\[
\sum_{i=1}^{n} \lambda_i^2 = 2|E| + |C|.
\] (1)

### 4.2 Theorem

Let G be a graph with n vertices, m edges, and let C be a minimum covering set of G. Then

\[ E_c(G) \leq \sqrt{n(2m + |C|)}. \]

**Proof**

Let \( \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_n \) be the eigenvalues of \( A_c(G) \). Then by Cauchy-Schwarz inequality,

\[
(\sum_{i=1}^{n} a_i b_i)^2 \leq (\sum_{i=1}^{n} a_i^2)(\sum_{i=1}^{n} b_i^2)
\]

We choose \( a_i = 1 \) and \( b_i = |\lambda_i| \) we get [by 1]

\[
E_c(G)^2 = \left( \sum_{i=1}^{n} |\lambda_i| \right)^2 \leq n \left( \sum_{i=1}^{n} |\lambda_i|^2 \right) = n \sum_{i=1}^{n} \lambda_i^2 = n(2m + |C|).
\]

\[ E_c(G)^2 = n(2m + |C|). \] (2)

### 4.3 Theorem

**Parity Theorem**

Let G be a graph with a minimum covering energy \( E_c(G) \) of G is a rational number, then

\[ E_c(G) \equiv |C| \pmod{2} \]

**Proof**

Let \( \lambda_1, \lambda_2, \ldots, \lambda_r \) be positive, and the rest of the minimum covering eigenvalues non-positive. Thus

\[
E_c(G) = \sum_{i=1}^{n} |\lambda_i| = (\lambda_1 + \lambda_2 + \ldots + \lambda_r) - (\lambda_{r+1} + \ldots + \lambda_n)
\]

Implying

\[
E_c(G) = 2(\lambda_1 + \lambda_2 + \ldots + \lambda_r) - |C|.
\]

Since \( \lambda_1, \lambda_2, \ldots, \lambda_r \) are algebraic integers, so is their sum. Hence \( (\lambda_1 + \lambda_2 + \ldots + \lambda_r) \) must be an integer if \( E_c(G) \) is rational.

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