International Journal of Recent Trends in Engineering & Research

ISSN (ONLINE) : 2455 – 1457 IMPACT FACTOR : 4.101

Observations on the hyberbola $y^2 = 150x^2 + 16$

Dr. G.Sumathi

Assistant Professor,Department of Mathematics,Shrimathi Indira Gandhi College,Trichy-600002.

Abstract : The binary quadratic equation $y^2 = 150x^2 + 16$ is considered and a few interesting properties among the solutions are presented.

Mathematics Subject Classification:11D09

Keywords: Binary quadratic,Integral solutions, Generalized Fibonacci Sequences, Generalized Lucas Sequences

I. NOTATIONS

$t_{m,n}$: Polygonal number of rank n with size m

P_n^m : Pyramidal number of rank n with size m

$\Pr_n$: Pronic number of rank n

S_n : Star number of rank n

$Ct_{m,n}$: Centered Pyramidal number of rank n with size m

$GF_n(k,s)$: Generalized Fibonacci Sequences

$GL_n(k,s)$: Generalized Lucas Sequences

II. INTRODUCTION

The binary quadratic equation of the form $y^2 = Dx^2 + 1$ where D is non-square positive integer has been studied by various mathematicians for its non-trivial integral solutions when D takes different integral values $[1,2,3,4]$.In [5] infinitely many Pythogorean triangles in each of which hypotenuse is four times the product of the generators added with unity are obtained by employing the non-integral solutions of binary quadratic equation $y^2 = 3x^2 + 1$.In [6],a special Pythogorean triangle is obtained by employing the integral solutions of $y^2 = 10x^2 + 1$. In [7],different patterns of infinitely many Pythogorean triangles are obtained by employing the non-integral solutions of $y^2 = 12x^2 + 1$.In this context one may also refer [8-14]. These results have motivated us to search for the integral solutions of yet another binary quadratic equation $y^2 = 150x^2 + 16$ representing a hyberbola. A few interesting properties among the solutions are presented.

III. METHOD OF ANALYSIS

The binary non-homogeneous quadratic diophantine equation represents a hyberbola to be solved for its non-zero integral solutions is

$$y^2 = 150x^2 + 16 \qquad (1)$$

whose initial solution is $x_0 = 16, 196$ (2)

To find the other solutions of (1),consider the pellian equation of (1) is given by

$$y^2 = 150x^2 + 1 \qquad (3)$$

whose general solution $(\tilde{x}_n, \tilde{y}_n)$ is represented by

$$\left.\begin{aligned} \tilde{x}_n &= \frac{g_n}{2\sqrt{150}} \\ \tilde{y}_n &= \frac{f_n}{2} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

where

$$\left.\begin{aligned} f_n &= \left[(49+4\sqrt{150})^{n+1} + (49+4\sqrt{150})^{n+1}\right] \\ g_n &= \left[(49+4\sqrt{150})^{n+1} - (49+4\sqrt{150})^{n+1}\right] \end{aligned}\right\}$$

where $n = 0,1,2,$

Employing the lemma of Brahamagupta between the solution (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$.The general solution of (1) is found to be

$$\left.\begin{aligned} x_{n+1} &= 8f_n + \frac{98}{\sqrt{150}} g_n \\ y_{n+1} &= 98f_n + 8\sqrt{150} g_n, n \geq 0,1,2,3.. \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

A few numerical examples are presented in the table below:

n	x_{n+1}	y_{n+1}
0	1568	19204
1	153648	1881796
2	15055936	184396804
3	1475328080	18069004996
4	144567091904	1770578092804
5	14166099678512	173498584089796
6	1388133201402272	17001090662707204
7	136022887637744144	1665933386361216196
8	13328854855297523840	163244470772736480004
9	1306091752931519592176	15996292202341813824196

A few interesting properties are given below:

1. $x_{n+1} \equiv 0 \pmod 4$
2. $y_{n+1} \equiv 0 \pmod 4$

3. **The recurrence relations satisfied by the values of** x_{n+1} **and** y_{n+1} **are respectively**

$$x_{n+3} - 98x_{n+2} + x_{n+1} = 0$$
$$y_{n+3} - 98y_{n+2} + y_{n+1} = 0$$

4.**Each of the following is a nasty number**

a) $6[9602x_{2n+4} - 940898x_{2n+3} + 32]$

b) $4[98x_{2n+3} - 9602x_{2n+2} + 32]$

c) $12[98y_{2n+2} - y_{2n+3} + 4]$

d) $6[76824y_{2n+3} - 784y_{2n+4} + 32]$

e) $1176[9603y_{2n+2} - y_{2n+4} + 392]$

f) $24[98y_{2n+2} - 1200x_{2n+2} + 8]$

g) $1176[98y_{2n+3} - 117600x_{2n+2} + 392]$

h) $24[y_{2n+4} - 9601x_{2n+2} + 32]$

i) $24[76824y_{2n+3} - 784x_{2n+3} + 32]$

j) $1176[9602y_{2n+2} - 1200x_{2n+3} + 392]$

k) $115224[940898y_{2n+2} - 1200x_{2n+4} + 38408]$

l) $1176[940898y_{2n+3} - 117600x_{2n+4} + 392]$

m) $1176[9602x_{2n+4} - 11523600x_{2n+3} + 392]$

5. **Each of the following is a cubical integer**

a) $(16)^2[98x_{3n+4} - 9602x_{3n+3} + 28806x_{3n+3} - 2822694x_{n+2}]$

b) $(16)^2[98x_{3n+4} - 9602x_{3n+3} + \frac{48}{16}[x_{n+3} - 9601x_{n+1}]]$

c) $(16)^2[98x_{3n+4} - 9602x_{3n+3} + \frac{48}{2}[98y_{n+1} - y_{n+2}]]$

d) $(16)^2[98x_{3n+4} - 9602x_{3n+3} + \frac{48}{16}[76824y_{n+2} - 784y]]$

e) $(196)^2[98y_{3n+4} - 117600x_{3n+3} + \frac{588}{19204}[98y_{n+3} - 11523600x_{n+1}]]$

f) $(196)^2[9603y_{3n+3} - y_{3n+5} + \frac{588}{4}[98y_{n+1} - 1200x_{n+1}]]$

g) $(4)^2[98y_{3n+3} - 1200x_{3n+3} + \frac{12}{196}[98y_{n+2} - 117600x_{n+1}]]$

h) $(16)^2[98x_{4n+5} - 9602x_{4n+4} + \frac{64}{16}[9602x_{n+3} - 940898x_{n+2}]]$

i) $(81)^2[171x_{3n+4} - 6489x_{3n+3} + \frac{243}{81}[492822y_{n+3} - 467532x_{n+3}]]$

6. Each of the following expression is a biquadratic integer:

a) $(16)^2\left[98x_{4n+5} - 9602x_{4n+4} + 64\left[\frac{9602x_{n+3} - 940898x_{n+2}}{16}\right]^2 - 32\right]$

b) $(4)^2\left[76824y_{4n+5} - 784y_{4n+6} + 64\left[\frac{98x_{n+2} - 9602x_{n+1}}{16}\right]^2 - 32\right]$

c) $(4)^2\left[x_{4n+6} - 9601x_{4n+4} + 64\left[\frac{98x_{n+2} - 9602x_{n+1}}{16}\right]^2 - 32\right]$

d) $(2)^2\left[98y_{4n+4} - y_{4n+5} + 8\left[\frac{98x_{n+2} - 9602x_{n+1}}{16}\right]^2 - 4\right]$

e) $(196)^2\left[9603y_{4n+4} - y_{4n+6} + 784\left[\frac{98x_{n+2} - 9602x_{n+1}}{16}\right]^2 - 392\right]$

f) $(2)^2\left[98y_{4n+4} - 1200x_{4n+4} + 16\left[\frac{98x_{n+2} - 9602x_{n+1}}{16}\right]^2 - 8\right]$

g) $(196)^2\left[98y_{4n+5} - 117600x_{4n+4} + 784\left[\frac{98x_{n+2} - 9602x_{n+1}}{16}\right]^2 - 392\right]$

h) $(2)^2\left[9602y_{4n+5} - 117600x_{4n+5} + 16\left[\frac{98x_{n+2} - 9602x_{n+1}}{16}\right]^2 - 8\right]$

i) $(19204)^2\left[98y_{4n+6} - 11523600x_{4n+4} + 76816\left[\frac{98x_{n+2} - 9602x_{n+1}}{16}\right]^2 - 38408\right]$

j) $(196)^2\left[9602y_{4n+6} - 11523600x_{4n+5} + 784\left[\frac{98x_{n+2} - 9602x_{n+1}}{16}\right]^2 - 392\right]$

k) $$(196)^2\left[9602y_{4n+4}-1200x_{4n+5}+784\left[\frac{98x_{n+2}-9602x_{n+1}}{16}\right]^2-392\right]$$

l) $$(196)^2\left[940898y_{4n+5}-117600x_{4n+6}+784\left[\frac{98x_{n+2}-9602x_{n+1}}{16}\right]^2-392\right]$$

m) $$(9)^2\left[171x_{4n+5}-6489x_{4n+4}+324\left[\frac{1443y_{n+1}-y_{n+3}}{81}\right]^2-162\right]$$

7.Each of the following expression is a quinticinteger:

a) $$(4)^3\left[98x_{5n+6}-9602x_{5n+5}+80\left(\frac{9602x_{n+3}-940898x_{n+2}}{16}\right)^3-80\left(\frac{9602x_{n+3}-940898x_{n+2}}{16}\right)\right]$$

b) $$(4)^3\left[98x_{5n+6}-9602x_{5n+5}+80\left(\frac{x_{n+3}-9601x_{n+1}}{16}\right)^3-80\left(\frac{x_{n+3}-9601x_{n+1}}{16}\right)\right]$$

c) $$(4)^3\left[9602x_{5n+7}-940898x_{5n+6}+10\left(\frac{98y_{n+1}-y_{n+2}}{2}\right)^3-10\left(\frac{98y_{n+1}-y_{n+2}}{2}\right)\right]$$

d) $$(2)^3\left[98y_{5n+5}-1200x_{5n+5}+20\left(\frac{98y_{n+1}-y_{n+2}}{2}\right)^3-20\left(\frac{98y_{n+1}-y_{n+2}}{2}\right)\right]$$

e) $$(14)^3\left[98y_{5n+6}-117600x_{5n+5}+980\left(\frac{76824y_{n+2}-784y_{n+3}}{16}\right)^3-980\left(\frac{76824y_{n+2}-784y_{n+3}}{16}\right)\right]$$

f) $$(2)^3\left[9602y_{n+6}-117600x_{5n+6}+20\left(\frac{98y_{n+1}-y_{n+2}}{2}\right)^3-20\left(\frac{98y_{n+1}-y_{n+2}}{2}\right)\right]$$

g) $$(4)^3\left[x_{5n+7}-9601x_{5n+5}+80\left(\frac{9602y_{n+2}-117600x_{n+2}}{4}\right)^3-80\left(\frac{9602y_{n+2}-117600x_{n+2}}{4}\right)\right]$$

h) $$(14)^3\left[9602y_{5n+7}-11523600x_{5n+6}+980\left(\frac{98y_{n+1}-1200x_{n+1}}{4}\right)^3-980\left(\frac{98y_{n+1}-1200x_{n+1}}{4}\right)\right]$$

i) $(14)^3\left[9602y_{5n+5}-12000x_{5n+6}+980\left(\frac{9602y_{n+2}-117600x_{n+2}}{4}\right)^3-980\left(\frac{9602y_{n+2}-117600x_{n+2}}{4}\right)\right]$

8.Remarkable observations:

I: Employing the linear combination among the solution of (1),one may generate integer solution for other choices of hyperbola which are presented in the table below:

S.NO	HYPERBOLA	(X_n, Y_n)
1	$Y_n^2-9600X_n^2=1024$	$X_n=98x_{n+1}-x_{n+2}$ $Y_n=98x_{n+2}-9602x_{n+1}$
2	$38416Y_n^2-38400X_n^2=39337984$	$X_n=9603x_{n+1}-x_{n+3}$ $Y_n=x_{n+3}-9601x_{n+1}$
3	$9604Y_n^2-150X_n^2=153664$	$X_n=9602x_{n+1}-8y_{n+2}$ $Y_n=1200x_{n+1}-y_{n+2}$
4	$Y_n^2-150X_n^2=256$	$X_n=38412x_{n+2}-392x_{n+3}$ $Y_n=4801x_{n+3}-470449x_{n+2}$
5	$Y_n^2-150X_n^2=64$	$X_n=9602x_{n+2}-784y_{n+2}$ $Y_n=9602y_{n+2}-117600x_{n+2}$
6	$Y_n^2-150X_n^2=153664$	$X_n=940898x_{n+2}-784y_{n+3}$ $Y_n=9602y_{n+3}-11523600x_{n+2}$
7	$Y_n^2-150X_n^2=1475174464$	$X_n=98x_{n+3}-76824y_{n+1}$ $Y_n=940898y_{n+1}-1200x_{n31}$
8	$Y_n^2-150X_n^2=153664$	$X_n=9602x_{n+3}-76824y_{n+2}$ $Y_n=940898y_{n+2}-117600x_{n+3}$
9	$9600Y_n^2-X_n^2=153600$	$X_n=98y_{n+2}-9602y_{n+1}$ $Y_n=98y_{n+1}-y_{n+2}$
10	$38400Y_n^2-38416X_n^2=1475174400$	$X_n=y_{n+3}-9601y_{n+1}$ $Y_n=9603y_{n+1}-y_{n+3}$

II: Employing the linear combination among the solution of (1),one may generate integer solution for other choices of parabola which are presented in the table below

S.NO	**PARABOLA**	(X_n, Y_n)
1	$Y_n - 300X_n^2 = 32$	$X_n = 98x_{n+1} - x_{n+2}$ $Y_n = 49x_{2n+3} - 4801x_{2n+2} + 16$
2	$2401Y_n - 150X_n^2 = 153664$	$X_n = 9603x_{n+1} - x_{n+3}$ $Y_n = x_{2n+4} - 9601x_{2n+2} + 32$
3	$19208Y_n - 150X_n^2 = 153664$	$X_n = 9602x_{n+1} - 8y_{n+2}$ $Y_n = y_{2n+3} - 1200x_{2n+2} + 4$
4	$8Y_n - 150X_n^2 = 256$	$X_n = 38412x_{n+2} - 392x_{n+3}$ $Y_n = 4801x_{2n+4} - 470449x_{2n+3} + 16$
5	$Y_n^2 - 7350X_n^2 = 784$	$X_n = x_{n+2} - 8y_{n+1}$ $Y_n = 9602y_{2n+2} - 1200x_{2n+3} + 392$
6	$8Y_n - 150X_n^2 = 64$	$X_n = 9602x_{n+2} - 784y_{n+2}$ $Y_n = 4801y_{2n+3} - 58800x_{2n+3} + 1$
7	$196Y_n - 150X_n^2 = 153664$	$X_n = 940898x_{n+2} - 784y_{n+3}$ $Y_n = 9602y_{2n+4} - 11523600x_{2n+3} + 392$
8	$9602Y_n - 75X_n^2 = 737587232$	$X_n = 98x_{n+3} - 76824y_{n+1}$ $Y_n = 940898y_{2n+2} - 1200x_{2n+4} + 38408$
9	$196Y_n - 150X_n^2 = 153664$	$X_n = 9602x_{n+3} - 76824y_{n+2}$ $Y_n = 940898y_{2n+3} - 117600x_{2n+4} + 392$
10	$19200Y_n = X_n^2 = 153600$	$X_n = 98y_{n+2} - 9602y_{n+1}$ $Y_n = 98y_{2n+2} - y_{2n+3} + 4$

III: Employingthe linear combination among the solution of (1),one may generate integer solution for other choices of straight line which are presented in the table below:

S.NO	**STRAIGHT LINE**	(X, Y)
1	$Y = 4X$	$X = \frac{1}{16}[98x_{n+2} - 9602x_{n+1}]$ $Y = 98y_{n+1} - 1200x_{n+1}$
2	$Y = 196X$	$X = \frac{1}{16}[9602x_{n+3} - 940898x_{n+2}]$ $Y = 98y_{n+2} - 117600x_{n+1}$

3	$Y = X$	$Y = 9602y_{n+1} - 1200x_{n+2}$ $X = 940898y_{n+2} - 117600x_{n+3}$
4	$Y = 19204X$	$Y = 98y_{n+3} - 11523600x_{n+1}$ $X = \frac{1}{16}[x_{n+3} - 9601x_{n+1}]$
5	$Y = 4X$	$Y = 9602y_{n+2} - 117600x_{n+2}$ $X = \frac{1}{2}98y_{n+1} - y_{n+2}$
6	$Y = 196X$	$Y = 9602y_{n+3} - 11523600x_{n+2}$ $X = \frac{1}{16}[76824y_{n+2} - 784y_{n+3}]$
7	$Y = 196X$	$Y = 9602y_{n+1} - 1200x_{n+2}$ $X = \frac{1}{196}[9603y_{n+1} - y_{n+3}]$
8	$Y = 19204X$	$Y = 940898y_{n+1} - 1200x_{n+3}$ $X = \frac{1}{16}[98x_{n+2} - 9602x_{n+1}]$
9	$Y = 196X$	$Y = 9602y_{n+3} - 11523600x_{n+2}$ $X = \frac{1}{19204}[940898y_{n+1} - 1200x_{n+3}]$
10	$Y = 2X$	$Y = 98y_{n+1} - y_{n+2}$ $X = \frac{1}{196}[9602y_{n+3} - 11523600x_{n+2}]$

IV: Employing the solutions of (1),each of the following among the special Polygonal,Pyramidal,Star number,Centered Pyramidal number and Pronic numbers is congruent to zero under modulo 16

(a) $\left(\frac{3P^3_{y-2}}{t_{3,y-2}}\right)^2 - 150\left(\frac{6P^4_{x-1}}{t_{3,2x-2}}\right)^2$

(b) $\left(\frac{P^5_y}{t_{3,y}}\right)^2 - 150\left(\frac{4P^5_x}{Ct_{4,x} - 1}\right)^2$

(c) $\left(\frac{18P^3_{y-2}}{Ct_{6,y-2} - 1}\right)^2 - 150\left(\frac{6P^5_{x-1}}{Ct_{6,x} - 1}\right)^2$

(d) $\left(\frac{6P^3_y}{\Pr_y}\right)^2 - 150\left(\frac{6P^5_x}{S_{x+1} - 1}\right)^2$

V: The solutions of (1) interms of special integers namely,Generalized Lucas GL_n and Fibonacci GF_n numbers are exhibited below:

$$\tilde{x}_n = \frac{GL_{n+1}}{2}(340,-1)$$

$$\tilde{y}_n = 39GF_{n+1}(340,-1)$$

IV. CONCLUSION

To conclude,one may search for other choices of positive pell equations for finding their integer solutions.

REFERENCES

I. Dickson L.E,History of Theory of numbers,Vol.2,Chelsea publishing company,Newyork,1952.
II. Mordel L.J, Diophantine Equations, Academic press,Newyork,1969.
III. Telang S.J,Number theory,Tata Mcgraw Hill Publishing Company Limited, New Delhi,2000.
IV. David Burton,Elementary Number Theory, Tata Mcgraw Hill Publishing Company Limited, New Delhi,2002.
V. Gopalan M.A and Janaki.G., Observation on $y^2 = 3x^2 + 1$,Acta Ciancia Indica,XXXIVM,No.2,693-696,2008.
VI. Gopalan M.A and Sangeetha .G, A Remarkable Observation on $Y^2 = 10X^2 + 1$,Impact Journal of Sciences and Technology,Vol,No.4,103- 106,(2010).
VII. Gopalan M.A,Palanikumar R., Observation on $Y^2 = 12X^2 + 1$,Antarctica J.Math 8(2),149-152,2011.
VIII. Gopalan M.A,Srividhya.G,Relations among M-gonal Number through the equation $Y^2 = 2X^2 + 1$, Antarctica J.Math., 7(3),363-369,2010.
IX. Gopalan M.A,Vijayasankarar R., Observation on the integral solutions of $Y^2 = 5X^2 + 1$,Impact Journal of Science and Technology,Vol.No.4,125-129,2010.
X. Gopalan M.A and Yamuna R.S, Remarkable Observation on the binary quadratic equation $Y^2 = (k^2 + 1)X^2 + 1, k \in Z - \{0\}$, Impact Journal of Science and Technology,Vol.No.4,61-65,2010.[ ]
XI. Gopalan M.A, and Sivagami B., Observation on the integral solutions of $Y^2 = 7X^2 + 1$, Antarctica J.Math ,7(3),291-296,2010.
XII. Gopalan M.A and Vijayalakshmi R.,Special Pythogorean triangles generated through the integral solutions of the equation $Y^2 = (k^2 - 1)X^2 + 1$, Antarctica Journal of Mathematics,795),503-507,2010.
XIII. Gopalan M.A., Vidhyalakshmi.S and Devibala .S,On the Diophantine Equation $3x^2 + xy = 14$,Acta Ciencia Indica,VolXXXIII M.No.2,645-648,2007.
XIV. Gopalan M.A., Vidhyalakshmi.S .,T.R.Usha rani.,and S.Mallika.,Observations on $y^2 = 12x^2 - 3$,Bessel J. Math ,2(3),153-158,2012.