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Slightly $\omega b-$ Continuous Functions in Topological Spaces

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Abstract—*In this paper, slightly $\omega b-$continuity is introduced and studied. Furthermore, basic properties and presentation theorems of slightly $\omega b-$continuous functions are investigated and relationships between slightly $\omega b-$continuous functions and graphs are studied and investigated*

Keywords—, Topological space, $\omega b-$*open set*, $\omega b-$*continuity*, slightly continuity, slightly $\omega b-$*continuity*.

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I. INTRODUCTION

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Semi–open sets, preopen sets, $\alpha-sets$, and $\beta-open$ sets play an important role in the researchers of generalizations of continuity in topological spaces. Indeed a significant theme in General Topology and Real Analysis concerns the variously modified forms of continuity, separation axioms etc. by utilizing generalized closed sets. Functions and of course continuous functions are stated among the most important and most researched points in the whole of the Mathematical Sciences. Many different forms of continuous functions have been introduced over the years. Various interesting problems arise when one considers continuity. Its importance is significant in various areas of mathematics and related sciences. Some of them are strongly $\alpha-irresoluteness$, $\alpha-irresoluteness$, $\alpha-continuity$, precontinuity, semi–continuity, $\gamma-continuity$ and slightly continuity. In 1980 Jain [9] introduced the notion of slightly continuous functions. In 1995 Nour [28] defined slightly semi–continuous functions as week form of slight continuity and investigated the functions. In 2000 Noiri and Chae [24] have further investigated slightly semi–continuous functions. On the other hand, Pal and Bhattacharyya [29] defined a function to be faintly precontinuous if the preimages of each clopen set of the codomain is preopen and obtained many properties of such functions. Slight continuity implies both slight semi-continuity and faint precontinuity but not conversely. In 2009 Noiri, Al-Omari and Noorani [27] introduced a new week form of both slightly and $\omega-continuous$, called slightly $\omega-continuous$, and studied basic properties and preservation theorems of slightly $\omega-continuous$ functions. In 2012 Chakraborty [4] introduced $b-continuous$ and studied the relations of slightly $b-continuous$ functions wih other forms of $b-continuous$ functions. In 2008 Noiri, Al-Omari and Noorani [26] introduced and investigated properties of a new generalization of class of $\omega-open$ set and $b-open$ set called $\omega b-open$ sets. The aim of this paper is to introduce and study a new weaker form of continuity called slightly $\omega b-continuity$. Moreover, basic properties and preservation theorems of slightly $\omega b-continuous$ functions are investigated and relationships between slightly $\omega b-continuous$

functions and graphs are investigated. In section 3, the notion of slightly $\omega b-continuous$ functions is also introduced and characterizations and some relationships of $\omega b-continuous$ functions and basic properties of slightly $\omega b-continuous$ functions are investigated and obtained. The relationships between slightly $\omega b-continuity$ and connectedness are investigated. In Section 4 and in Section 5, the relationships between slightly $\omega b-continuity$ and compactness and the relationships between slightly $\omega b-continuity$ and separation axioms and graphs are obtained.

II. PRELIMINARIES

Throughout this paper, (X,τ) and (Y,σ) (or simply, X and Y) denote topological spaces on which no separation axioms are assumed unless explicitly stated. For a subset A of a space (X,τ), $Cl(A)$, $Int(A)$ and $X-A$ denote the closure of A, the interior of A and the complement of A in X, respectively. Recently, as generalization of closed sets, the notion of $\omega-$closed sets were introduced and studied by Hdeib [7]. Let (X,τ) be a topological space and let A be a subset of X. A point $x\in X$ is called a condensation point of A if for each $U\in\tau$ with $x\in U$, the set $U\cap A$ is uncountable. A subset A is said to be $\omega-$closed [7] if it contains all its condensation points. The complement of an $\omega-$closed sets is said to be an $\omega-$open set. It is well known that a subset W of a space (X,τ) is $\omega-$open if and only if for each $x\in W$, there exists $U\in\tau$ such that $x\in U$ and $U-W$ is countable. The family of all $\omega-$open subsets of a topological space (X,τ), denoted by $\omega O(X,\tau)$, forms a topology on X which is finer than τ. The set of all $\omega-$open sets of (X,τ) containing a point $x\in X$ is denoted by $\omega O(X,x)$. The complement of an $\omega-$open set is said to be $\omega-closed$. The intersection of all $\omega-closed$ sets of X containing A is called the closure of A and is denoted by $\omega Cl(A)$. The union of all $\omega-open$ sets of X contained in A is called $\omega-\text{int}erior$ of A and is denoted by $\omega Int(A)$. The family of all $\omega-open$ open, $\omega-$closed, clopen, $\omega-$clopen sets of X is denoted by $\omega O(X,\tau)$, $\omega Cl(X,\tau)$, $CO(X,\tau)$, $\omega CO(X,\tau)$.

A subset A of a topological space X is said to be $b-open$ [1] if $A\subseteq Int\left[Cl(A)\right]\cup Cl\left[Int(A)\right]$. The complement of a $b-open$ set is called $b-closed$. The intersection of all $b-closed$ sets of X containing A is called the $b-closure$ of A and is denoted by $bCl(A)$. The union of all $b-open$ sets of X contained in A is called the $b-\text{int}erior$ of A and is denoted by $bInt(A)$. A subset A of X is said to be regular open if $A=Int\left[Cl(A)\right]$. The family of all $b-open$ (resp. $b-closed$, clopen, $b-clopen$, regular open) sets in X is denoted by $BO(X)$ (resp. $BC(X)$, $CO(X)$, $BCO(X)$, $RO(X)$).

DEFINITION 2.1. A subset A of a space X is said to be $\omega b-open$ if for every $x\in A$, there exists a $b-open$ subset $U_x\subseteq X$ containing x such that U_x-A is countable. The complement of an $\omega b-open$ subset is said to be $\omega b-closed$.

The family of all $\omega b-open$ sets in a topological space (X,τ) is denoted by $\omega b-O(X,\tau)$ or $\omega b-O(X)$. The family of all $\omega b-closed$ subsets in a topological space (X,τ) is denoted by $\omega b-C(X,\tau)$ or $\omega b-C(X)$.

For any $x\in X$, we present $\omega b-O(X,x)=\{U\subseteq X : x\in U \text{ and } U \text{ is } \omega b-open \text{ in } X\}$.

LEMMA 2.2. For a subset of a topological space, both $\omega-openness$ and $b-openness$ imply $\omega b-openness$.

Proof. (1) Assume A is $\omega-open$, then for each $x \in A$, there is an open set U_x containing x such that $(U_x - A)$ is countable set. Since every open set is $b-open$, A is $\omega b-open$.

(2) Let A be $b-open$. For each $x \in A$, there exists a $b-open$ set $U_x = A$ such that $x \in U_x$ and $U_x - A = \phi$. Therefore, A is $\omega b-open$.

The following diagram shows the implications for properties of subsets

$$\begin{array}{ccc} open\ set & \longrightarrow & b-open\ set \\ \downarrow & & \downarrow \\ \omega-open\ set & \longrightarrow & \omega b-open\ set \end{array}$$

The converses need not be true as shown by the examples 2.3 and 2.4 in [26].

LEMMA 2.3. [26]. A subset A of a space X is $\omega b-open$ if and only if for every $x \in A$, there exists a $b-open$ subset U containing x and a countable subset C such that $U - C \subseteq A$.

The intersection of two $\omega b-open$ sets is not always $\omega b-open$.

EXAMPLE 2.4. Let $X = R$ with the usual topology τ. Let $A = Q$ be the set of all rational numbers and $B = [0,1)$. Then A and B are $\omega b-open$, but $A \cap B$ is not $\omega b-open$, since each $b-open$ set containing 0 is uncountable set.

PROPOSITION 2.5. The union of any family of $\omega b-open$ sets is $\omega b-open$.

PROOF. Let $\{A_\alpha : \alpha \in \Delta\}$ be a collection of $\omega b-open$ subsets of X, then for every $x \in \bigcup_{\alpha \in \Delta} A_\alpha$, $x \in A_\beta$ for some $\beta \in \Delta$. Hence there exists a $b-open$ subset U of X containing x such that $(U - A_\beta)$ is countable. Now as $U - (\bigcup_{\alpha \in \Delta} A_\alpha) \subseteq (U - A_\beta)$ and thus $U - (\bigcup_{\alpha \in \Delta} A_\alpha)$ is countable. Therefore, $\bigcup_{\alpha \in \Delta} A_\alpha$ is $\omega b-open$.

The intersection of all $\omega b-closed$ sets of X containing A is called the $\omega b-closure$ of A and is denoted by $\omega b-Cl(A)$. The union of all $\omega b-open$ sets of X contained in A is called the $\omega b-interior$ of A and is denoted by $\omega b-Int(A)$.

PROPOSITION. 2.6. [26]. The intersection of an $\omega b-open$ set with an open set is $\omega b-open$.

III. SLIGHTLY $\omega b-$ CONTINUOUS FUNCTIONS

In this section, the notion of slightly $\omega b-continuous$ functions is introduced and characterizations and some relationships of $\omega b-continuous$ functions and basic properties of slightly $\omega b-continuous$ functions are investigated and obtained.

DEFINITION 3.1. A function $f:(X,\tau) \longrightarrow (Y,\sigma)$ is called slightly $\omega b-continuous$ at a point $x \in X$ if for each clopen subset V in Y containing $f(x)$, there exists an $\omega b-open$ subset U in X containing x such that $f(U) \subseteq V$.

DEFINITION 3.2. A function $f:(X,\tau) \longrightarrow (Y,\sigma)$ is called slightly $\omega b-continuous$ if it is slightly $\omega b-continuous$ at each point of X.

THEOREM 3.3. Let (X,τ) and (Y,σ) be topological spaces and let $f:(X,\tau) \longrightarrow (Y,\sigma)$ be a function. Then the following statements are equivalent.

(1) f is slightly $\omega b-continuous$;

(2) for every clopen set $V \subseteq Y$, $f^{-1}(V)$ is $\omega b-open$;

(3) for every clopen set $V \subseteq Y$, $f^{-1}(V)$ is $\omega b-closed$;

(4) for every clopen set $V \subseteq Y$, $f^{-1}(V)$ is $\omega b-clopen$.

PROOF. $(1)\Rightarrow(2)$: Let V be a clopen subset of Y and let $x \in f^{-1}(V)$. Since $f(x)\in V$, by (1), there exists an $\omega b-open$ set U_x in X such that $x \in U_x$ and $U_x \subseteq f^{-1}(V)$. We obtain that $f^{-1}(V) = \cup\{U_x : x \in f^{-1}(V)\}$. Thus, $f^{-1}(V)$ is $\omega b-open$.

$(2)\Rightarrow(3)$: Let V be a clopen subset of Y. Then, $(Y-V)$ is clopen. By (2), $f^{-1}(Y-V) = X - f^{-1}(V)$ is an $\omega b-open$ set in X. Thus, $f^{-1}(V)$ is $\omega b-closed$.

$(3)\Rightarrow(4)$: Let V be a clopen subset of Y. Then, by (3), $f^{-1}(V)$ is an $\omega b-closed$ set in X. Note that $Y-V$ is also clopen in Y. Hence by (3), it follows that $f^{-1}(Y-V) = X - f^{-1}(V)$ is a $\omega b-closed$ set in X. Thus, $f^{-1}(V)$ is $\omega b-clopen$ in X.

$(4)\Rightarrow(1)$: Let V be a clopen subset of Y containing $f(x)$. By (4), $f^{-1}(V)$ is $\omega b-clopen$ in X. Take $U = f^{-1}(V)$. Then, $f(U)\subseteq V$. Hence, f is slightly $\omega b-continuous$.

THEOREM. 3.4. Let $f:(X,\tau)\longrightarrow(Y,\sigma)$ be a function and $\Sigma=\{U_i : i\in I\}$ be a cover of X such that $U_i \in \omega b-O(X,\tau)$ for each $i \in I$. If $f|U_i$ is slightly $\omega b-continuous$ for each $i\in I$, then f is a slightly $\omega b-continuous$ function.

PROOF. Suppose that V is any clopen set of Y. Since $f|U_i$ is slightly $\omega b-continuous$ for each $i\in I$, it follows that $(f|U_i)^{-1}(V)\in \omega b-O(U_i,\tau|U_i)$. We have $f^{-1}(V)=\cup\{f^{-1}(V)\cap U_i : i\in I\}$ $=\cup\{(f|U_i)^{-1}(V) : i\in I\}$. We obtain $(f)^{-1}(V)\in \omega b-O(X,\tau)$ which means that f is slightly $\omega b-continuous$.

THEOREM 3.5. Let $f:(X,\tau)\longrightarrow(Y,\sigma)$ be a function and let $g: X\longrightarrow X\times Y$ be the graph function of f, defined by $g(x)=(x,f(x))$ for every $x\in X$. Then g is slightly $\omega b-continuous$ if and only if f is slightly $\omega b-continuous$.

PROOF. Let $V\in CO(Y,\sigma)$. Then $X\times V \in CO(X\times Y)$. Since g is slightly $\omega b-continuous$, then $f^{-1}(V)=g^{-1}(X\times V)\in \omega b-O(X,\tau)$. Thus, f is slightly $\omega b-continuous$.

Conversely, let $x\in X$ and let W be a clopen subset of $X\times Y$ containing $g(x)$. Then $W\cap(\{x\}\times Y)$ is clopen in $\{x\}\times Y$ containing $g(x)$. Also $\{x\}\times Y$ is homeomorphic to Y. Hence $\{y\in Y:(x,y)\in W\}$ is a clopen subset of Y. Since f is slightly $\omega b-continuous$, $\cup\{f^{-1}(y):(x,y)\in W\}$ is an $\omega b-open$ subset of X. Further $x\in\cup\{f^{-1}(y):(x,y)\in W\}\subseteq g^{-1}(W)$. Hence $g^{-1}(W)$ is $\omega b-open$. Then g is slightly $\omega b-continuous$.

DEFINITION 3.6. A function $f:(X,\tau)\longrightarrow(Y,\sigma)$ is called $\omega b-irresolute$ if for every $\omega b-open$ subset G of Y, $f^{-1}(G)$ is $\omega b-open$ in X.

DEFINITION 3.7. A function $f:(X,\tau)\longrightarrow(Y,\sigma)$ is called $\omega b-open$ if for every $\omega b-open$ subset A of X, $f(A)$ is $\omega b-open$ in Y.

DEFINITION 3.8. A function $f:(X,\tau)\longrightarrow(Y,\sigma)$ is said to be $\omega b-continuous$ if $f^{-1}(V)$ is $\omega b-open$ set in X for each open set V of Y.

DEFINITION 3.9. A function $f:(X,\tau)\longrightarrow(Y,\sigma)$ is slightly continuous if $f^{-1}(V)$ is open set in X for each clopen set V of Y.

THEOREM 3.10. Let $f:(X,\tau)\longrightarrow(Y,\sigma)$ and $g:(Y,\sigma)\longrightarrow(Z,\delta)$ be functions. Then, the following properties hold:

(1) If f is $\omega b-irresolute$ and g is slightly $\omega b-continuous$, then $gof:(X,\tau)\longrightarrow(Z,\delta)$ is slightly $\omega b-continuous$.

(2) If f is $\omega b-irresolute$ and g is $\omega b-continuous$, then $gof:(X,\tau)\longrightarrow(Z,\delta)$ is slightly $\omega b-continuous$.

(3) If f is $\omega b-irresolute$ and g is slightly continuous, then $gof:(X,\tau)\longrightarrow(Z,\delta)$ is slightly $\omega b-continuous$.

PROOF. (1) Let V be any clopen set in Z. Since g is slightly $\omega b-continuous$, $g^{-1}(V)$ is $\omega b-open$, Since f is $\omega b-irresolute$, $f^{-1}\left[g^{-1}(V)\right]=(gof)^{-1}(V)$ is $\omega b-open$. Therefore gof is slightly $\omega b-continuous$.

(2) and (3) can be obtained similarly.

THEOREM 3.11. Let $f:(X,\tau)\longrightarrow(Y,\sigma)$ and $g:(Y,\sigma)\longrightarrow(Z,\delta)$ be functions. If f is $\omega b-open$ and surjective and $gof:X\longrightarrow Z$ is slightly $\omega b-continuous$, then g is slightly $\omega b-continuous$.

PROOF. Let V be any clopen set in Z. Since gof is slightly $\omega b-continuous$, $(gof)^{-1}(V)=f^{-1}\left[g^{-1}(V)\right]$ is $\omega b-open$. Since f is $\omega b-open$, then $f\left(f^{-1}\left[g^{-1}(V)\right]\right)=g^{-1}(V)$ is $\omega b-open$. Hence, g is slightly $\omega b-continuous$.

Combining the previous two theorems, we obtain the following result.

THEOREM 3.12. Let $f:(X,\tau)\longrightarrow(Y,\sigma)$ be surjective, $\omega b-irresolute$ and $\omega b-open$ and $g:(Y,\sigma)\longrightarrow(Z,\delta)$ be a function. Then $gof:(X,\tau)\longrightarrow(Z,\delta)$ is slightly $\omega b-continuous$ if and only if g is slightly $\omega b-continuous$.

DEFINITION. 3.13. Let (X,τ) be a topological space. Then a filter base Λ is said to be $\omega b-convergent$ to a point $x\in X$ if for any $U\in\omega b-O(X,\tau)$ containing x, there exists a $B\in\Lambda$ such that $B\subseteq U$.

DEFINITION 3.14. Let (X,τ) be a topological space. A filter base Λ is said to be $co-convergent$ to a point x in X if for any $U\in CO(X,\tau)$ containing x, there exists a $B\in\Lambda$ such that $B\subseteq U$.

THEOREM 3.15. If a function $f:(X,\tau)\longrightarrow(Y,\sigma)$ is slightly $\omega b-continuous$, then for each point $x\in X$ and each filter base Λ in X $\omega b-converging$ to x, the filter base $f(\Lambda)$ is $co-convergent$ to $f(x)$.

PROOF. Let $x \in X$ and Λ be any filter base in X $\omega b-converging$ to x. Since f is slightly $\omega b-continuous$, then for any $V \in CO(Y,\sigma)$ containing $f(x)$, there exists a $U \in \omega b-O(X,\tau)$ containing x such that $f(U) \subseteq V$. Since Λ is $\omega b-converging$ to x, there exists a $B \in \Lambda$ such that $B \subseteq U$. This means that $f(B) \subseteq V$ and therefore the filter base $f(\Lambda)$ is $co-convergent$ to $f(x)$.

DEFINITION 3.16. A topological space (X,τ) is called $\omega b-connected$ provided that X is not the union of two disjoint nonempty $\omega b-open$ sets.

THEOREM 3.17. If a function $f:(X,\tau)\longrightarrow(Y,\sigma)$ is slightly $\omega b-continuous$ surjective function and X is $\omega b-connected$ space, then Y is connected space.

PROOF. Suppose that Y is not connected space. Then there exists nonempty disjoint open sets U and V such that $Y = U \cup V$. Therefore, U and V are clopen sets in Y. Since f is slightly $\omega b-continuous$, then $f^{-1}(U)$ and $f^{-1}(V)$ are $\omega b-closed$ and $\omega b-open$ in X. Moreover, $f^{-1}(U)$ and $f^{-1}(V)$ are nonempty disjoint and $X = f^{-1}(U) \cup f^{-1}(V)$. This shows that X is not $\omega b-connected$. This is a contradiction. Hence, Y is connected.

DEFINITION 3.18. A topological space (X,τ) is called $0-\dim ensional$ if its topology has a base consisting of clopen sets.

THEOREM 3.19. Suppose that $f:(X,\tau)\longrightarrow(Y,\sigma)$ is slightly $\omega b-continuous$ and Y is $0-\dim ensional$ space, then f is $\omega b-continuous$.

PROOF. Let $x \in X$ and V be any open subset of Y containing $f(x)$. Since Y is a $0-\dim ensional$ space, there exists a clopen set U containing $f(x)$ such that $U \subseteq V$. Since f is slightly $\omega b-continuous$, then there exists an $\omega b-open$ subset G in X containing x such that $f(G) \subseteq U \subseteq V$. Thus, f is $\omega b-continuous$.

DEFINITION 3.20. A subset M of a topological space (X,τ) is said to be $\omega b-dense$ in X if there is no proper $\omega b-closed$ set C in X such that $M \subseteq C \subseteq X$.

PROPOSITION 3.21. A subset M of a topological space (X,τ) is said to be $\omega b-dense$ in X if for any nonempty $\omega b-open$ set U in X, $U \cap M \neq \phi$.

THEOREM 3.22. Let $f:(X,\tau)\longrightarrow(Y,\sigma)$ be a surjective function and (Y,σ) be $0-\dim ensional$ *space*. Then the following statements are equivalent:

(1) f is slightly $\omega b-continuous$.

(2) If C is a clopen subset of Y such that $f^{-1}(C) \neq X$, then there is a proper $\omega b-closed$ subset D of X such that $f^{-1}(C) \subseteq D$.

(3) If M is an $\omega b-dense$ subset of X, then $f(M)$ is a dense subset of Y.

PROOF. $(1)\Rightarrow(2)$: Let C be a clopen subset of Y such that $f^{-1}(C) \neq X$. Then $(Y-C)$ is a clopen set in Y such that $f^{-1}(Y \setminus C) = X \setminus f^{-1}(C) \neq \phi$. By (1), there exists an $\omega b-open$ set V in Y such that $V \neq \phi$ and $V \subseteq f^{-1}(Y \setminus C) = X \setminus f^{-1}(C)$. This shows that $f^{-1}(C) \subseteq (X \setminus V)$ and $X \setminus V = D$ is a proper $\omega b-closed$ set in X.

$(2)\Rightarrow(3)$: Let M be an $\omega b-dense$ set in X. Suppose that $f(M)$ is not dense in Y. Then there exists a nonempty proper closed set C in Y such that $f(M) \subseteq C \subset Y$. Clearly $f^{-1}(C) \neq X$. Then

$U = Y \setminus C$ is a nonempty proper open subset of Y. Since Y is $0-\dim ensional$. So there exists a nonempty proper clopen set E in Y such that $E \subseteq U = Y \setminus C$. Then $C \subseteq Y \setminus E = F$ and $F = Y - E$ is a nonempty proper clopen set in Y. Also $f(M) \subseteq F \subseteq Y$ and $f^{-1}(F) \neq \phi$. By (2), there exists a nonempty proper closed set D such that $M \subseteq f^{-1}(F) \subseteq D \subset X$. This is a contradiction to the fact that M is $\omega b-dense$ in X.

$(3) \Rightarrow (1)$: Suppose that f is not a slightly $\omega b-continuous$ function, then there exits a nonempty proper clopen set U in Y such that $\omega b-\text{int}erior$ of $f^{-1}(U)$ is empty, that is $X - f^{-1}(U)$ is $\omega b-dense$ in X, while $f(X - f^{-1}(U)) = Y - U$ is not dense in Y. This is a contradiction.

PROPOSITION 3.23. Let $f:(X,\tau) \longrightarrow (Y,\sigma)$ be a function and $X = A \cup B$, where $A, B \in \tau$. If the restriction functions $f_{|A}:(A,\tau_{|A}) \longrightarrow (Y,\sigma)$ and $f_{|B}:(B,\tau_{|B}) \longrightarrow (Y,\sigma)$ are slightly $\omega b-continuous$, then f is slightly $\omega b-continuous$.

PROOF. Let $x \in X$ and let U be any clopen subset of Y such that $f(x) \in U$. Now $x \in f^{-1}(U)$. Then $x \in (f_{|A})^{-1}(U)$ or $x \in (f_{|B})^{-1}(U)$ or both $x \in (f_{|A})^{-1}(U)$ and $x \in (f_{|B})^{-1}(U)$. Suppose $x \in (f_{|A})^{-1}(U)$. Since $f_{|A}$ is slightly $\omega b-continuous$, there exists an $\omega b-open$ set V in A such that $x \in V$ and $x \in V \subseteq (f_{|A})^{-1}(U) \subseteq f^{-1}(U)$. Since V is $\omega b-open$ in A and A is open in X. Hence V is $\omega b-open$ in X. Thus we find that f is slightly $\omega b-continuous$.

IV. COVERING PROPERTIES

In this section, the relationship between slightly $\omega b-continuous$ functions and compactness are investigated.

DEFINITION 4.1. A topological space (X,τ) is said to be mildly compact if for every clopen cover of X has a finite subcover.

DEFINITION 4.2. A topological space (X,τ) is said to be $\omega b-compact$ if for every $\omega b-open$ cover of X has a finite subcover.

DEFINITION 4.3. Let (X,τ) be a topological space. Then a subset A of X is said to be mildly compact (respectively $\omega b-compact$) relative to X if every cover of A by clopen (resp. $\omega b-open$) sets of X has a finite subcover.

DEFINITION. 4.4. A subset A of a space X is said to be mildly compact (respectively $\omega b-compact$) if the subspace A is mildly compact (resp. $\omega b-compact$).

THEOREM. 4.5. If a function $f:(X,\tau) \longrightarrow (Y,\sigma)$ is slightly $\omega b-continuous$ and K is $\omega b-compact$ relative to X, then $f(K)$ is mildly compact in Y.

PROOF. Let $\{H_\alpha : \alpha \in I\}$ be any cover of $f(K)$ by clopen sets of the subspace $f(K)$. For each $\alpha \in I$, there exists a clopen set K_α of Y such that $H_\alpha = K_\alpha \cap f(K)$. For each $x \in K$, there exists $\alpha_x \in I$, such that $f(x) \in K_{\alpha_x}$ and there exists $U_x \in \omega b-O(X,\tau)$ containing x such that $f(U_x) \subseteq K_{\alpha_x}$. Since the family $\{U_x : x \in K\}$ is a cover of K by $\omega b-open$ sets of K, there exists a finite subset K_0 of K such that $K \subseteq \cup\{U_x : x \in K_0\}$. Therefore, we obtain

$f(K)\subseteq \cup\{f(U_x): x\in K_0\}$ which is a subset of $\cup\{K_{\alpha_x}: x\in K_0\}$. Thus $f(K)=\cup\{H_{\alpha_x}: x\in K_0\}$ and hence $f(K)$ is mildly compact.

COROLLARY 4.6. If $f:(X,\tau)\longrightarrow(Y,\sigma)$ is slightly $\omega b-continuous$ surjection and X is $\omega b-compact$, then Y is mildly compact.

DEFINITION 4.7. A topological space (X,τ) said to be mildly countably compact if every clopen countable cover of X has a finite subcover.

DEFINITION 4.8. A topological space (X,τ) is said to be mildly Lindelof if every cover of X by clopen sets has a countable subcover.

DDEFINITION 4.9. A topological space (X,τ) said to be countably $\omega b-compact$ if every countable $\omega b-open$ cover of X has a finite subcover.

DEFINTION 4.10. A topological space (X,τ) said to be $\omega b-Lindeloff$ if every $\omega b-open$ cover of X has a countable subcover.

DEFINTION 4.11. A topological space (X,τ) said to be $\omega b-closed-compact$ if every $\omega b-closed$ cover of X has a finite subcover.

DEFINTION 4.12. A topological space (X,τ) said to be countably $\omega b-closed-compact$ if every countable cover of X by $\omega b-closed$ sets has a finite subcover.

DEFINTION 4.13. A topological space (X,τ) said to be $\omega b-closed-Lindeloff$ if every cover of X by $\omega b-closed$ sets has a countable subcover.

THEOREM 4.14. Let (X,τ) be $\omega b-Lindeloff$ and let $f:(X,\tau)\longrightarrow(Y,\sigma)$ be a slightly $\omega b-continuous$ surjection. Then Y is mildly Lindelof.

PROOF. Let $\gamma=\{V_\alpha:\alpha\in I\}$ be any clopen cover of Y. Since f is slightly $\omega b-continuous$, then $\lambda=f^{-1}(\gamma)=\{f^{-1}(V_\alpha):\alpha\in I\}$ is an $\omega b-open$ cover of X. Since X is $\omega b-Lindeloff$, there exists a countable subset I_0 of I such that $X=\cup\{f^{-1}(V_\alpha):\alpha\in I_0\}$. Thus, we have $Y=\cup\{V_\alpha:\alpha\in I_0\}$ and Y is mildly Lindelof.

THEOREM 4.15. Let (X,τ) be countably $\omega b-compact$ and let $f:(X,\tau)\longrightarrow(Y,\sigma)$ be a slightly $\omega b-continuous$ surjection. Then Y is mildly countably compact.

PROOF. Let $\gamma=\{V_\alpha:\alpha\in I\}$ be a countable open cover of X . Since f is slightly $\omega b-continuous$, then $\lambda=f^{-1}(\gamma)=\{f^{-1}(V_\alpha):\alpha\in I\}$ is a countable $\omega b-open$ cover of X . Since X is countably $\omega b-compact$, there exists a finite subset I_0 of I such that $X=\cup\{f^{-1}(V_\alpha):\alpha\in I_0\}$. Thus, we have $Y=\cup\{V_\alpha:\alpha\in I_0\}$ and Y is mildly countably compact.

The proofs of the next three theorems can be obtained similarly as the previous two theorems.

THEOREM 4.16. Let $f:(X,\tau)\longrightarrow(Y,\sigma)$ be a slightly $\omega b-continuous$ surjection. Let (X,τ) be $\omega b-closed$ compact, and Then Y is mildly compact.

THEOREM 4.17. Let (X,τ) be $\omega b-closed-Lindeloff$ and let $f:(X,\tau)\longrightarrow(Y,\sigma)$ be a slightly $\omega b-continuous$ surjection. Then Y is mildly Lindelof.

THEOREM 4.18. Let (X,τ) be countably $\omega b-closed-compact$ and let $f:(X,\tau)\longrightarrow(Y,\sigma)$ be a slightly $\omega b-continuous$ surjection. Then Y is mildly countably compact.

V. SEPARATION AXIOMS

In this section, the relationships between slightly $\omega b-continuous$ functions and separation axioms are investigated.

DEFINITION 5.1. A topological space (X,τ) said to be $\omega b-T_1$ if for each pair of distinct points x and y of X, there exist $\omega b-open$ sets U and V containing x and y respectively such that $y\notin U$ and $x\notin V$.

DEFINITION 5.2. A topological space (X,τ) said to be $\omega b-T_2$ $(\omega b-Hausdorff)$ if for each pair of distinct points x and y in X, there exist disjoint $\omega b-open$ sets U and V in X such that $x\in U$ and $y\in V$.

DEFINITION 5.3. A topological space (X,τ) said to be clopen T_1 if for each pair of distinct points x and y of X, there exist clopen sets U and V containing x and y respectively such that $y\notin U$ and $x\notin V$.

DEFINITION 5.4. A topological space (X,τ) said to be clopen T_2 (*clopen Hausdorff or ultra Hausdorff*) if for each pair of distinct points x and y in X, there exist disjoint clopen sets U and V in X such that $x\in U$ and $y\in V$.

PROPOSITION 5.5. A topological space (X,τ) is $\omega b-T_1$ if and only if the singletons are $\omega b-closed$ sets.

PROPOSITION 5.6. A topological space (X,τ) is $\omega b-T_2$ $(\omega b-Hausdorff)$ if and only if the intersection of all $\omega b-closed$ $\omega b-neighbourhoods$ of each point of X is reduced to that point.

THEOREM 5.7. If a function $f:(X,\tau)\longrightarrow(Y,\sigma)$ is slightly $\omega b-continuous$ injection and Y is clopen T_1, then X is $\omega b-T_1$.

PROOF. Suppose that Y is T_1. For any distinct points x and y in X, there exist $V,W\in CO(Y,\sigma)$ such that $f(x)\in V$, $f(y)\notin V$, $f(x)\notin W$ and $f(y)\in W$. Since f is slightly $\omega b-continuous$, $f^{-1}(V)$ and $f^{-1}(W)$ are $\omega b-open$ subsets of X such that $x\in f^{-1}(V)$, $y\notin f^{-1}(V)$, $x\notin f^{-1}(W)$ and $y\in f^{-1}(W)$. This shows that X is $\omega b-T_1$.

THEOREM 5.8. If $f:(X,\tau)\longrightarrow(Y,\sigma)$ is a slightly $\omega b-continuous$ injection and Y is clopen T_2, then X is $\omega b-T_2$.

PROOF. For any pair of distinct points x and y in X, there exist disjoint clopen sets U and V in Y such that $f(x)\in U$ and $f(y)\in V$. Since f is slightly $\omega b-continuous$, $f^{-1}(U)$ and $f^{-1}(V)$ are $\omega b-open$ in X containing x and y respectively. Therefore $f^{-1}(U)\cap f^{-1}(V)=\phi$ because $U\cap V=\phi$. This shows that X is $\omega b-T_2$.

THEOREM 5.9. If $f:(X,\tau)\longrightarrow(Y,\sigma)$ is a slightly continuous function and $g:(X,\tau)\longrightarrow(Y,\sigma)$ is slightly $\omega b-continuous$ function and Y is clopen Hausdorff, then $E=\{x\in X: f(x)=g(x)\}$ is $\omega b-clsed$ in X.

PROOF. If $x \in (X - E)$, then it follows that $f(x) \neq g(x)$. Since Y is clopen Hausdorff, there exist $f(x) \in V \in CO(Y,\sigma)$ and $g(x) \in W \in CO(Y,\sigma)$ such that $V \cap W = \phi$. Since f is slightly continuous and g is slightly $\omega b-continuous$, then $f^{-1}(V)$ is open and $g^{-1}(W)$ is $\omega b-open$ in X with $x \in f^{-1}(V)$ and $x \in g^{-1}(W)$. Set $O = f^{-1}(V) \cap g^{-1}(W)$. By Corollary 2.9 of [26], O is $\omega b-open$. We notice that $x \in O$ and $f(O) \cap g(O) = \phi$. It follows that $O \cap E = \phi$. Thus it shows that $x \notin \omega b-Cl(E)$. This proves that E is $\omega b-closed$ in X.

DEFINITION 5.10. Let $f:(X,\tau) \longrightarrow (Y,\sigma)$ be a function. Then the subset $\{(x, f(x)) : x \in X\} \subseteq X \times Y$ is called the graph of f and is denoted by $G(f)$.

DEFINITION 5.11. A graph $G(f) = \{(x, f(x)) : x \in X\}$ of a function $f:(X,\tau) \longrightarrow (Y,\sigma)$ is said to be strongly $\omega b-co-closed$ if for each $(x,y) \in (X \times Y) \setminus G(f)$, there exist $U \in \omega b-CO(X,\tau)$ containing x and $V \in CO(Y,\sigma)$ containing y such that $(U \times V) \cap G(f) = \phi$.

LEMMA 5.12. A graph $G(f) = \{(x, f(x)) : x \in X\}$ of a function $f:(X,\tau) \longrightarrow (Y,\sigma)$ is strongly $\omega b-co-closed$ in $X \times Y$ if and only if for each $(x,y) \in [(X \times Y) - G(f)]$, there exist $U \in \omega b-CO(X,\tau)$ containing x and $V \in CO(Y,\sigma)$ containing y such that $f(U) \cap V = \phi$.

THEOREM 5.13. If $f:(X,\tau) \longrightarrow (Y,\sigma)$ is slightly $\omega b-continuous$ function and Y is clopen T_1, then $G(f)$ is strongly $\omega b-co-closed$ in $X \times Y$.

PROOF. Let $(x,y) \in (X \times Y) \setminus G(f)$, then $f(x) \neq y$ and there exists a clopen set V of Y such that $f(x) \in V$ and $y \notin V$. Since f is slightly $\omega b-continuous$, then $f^{-1}(V) \in \omega b-CO(X,\tau)$ containing x. Take $U = f^{-1}(V)$. We have $f(U) \subseteq V$. Therefore, we obtain $f(U) \cap (Y - V) = \phi$ and $(Y - V) \in CO(Y,\sigma)$ containing y. This shows that $G(f)$ is strongly $\omega b-co-closed$ in $X \times Y$.

COROLLARY 5.14. If $f:(X,\tau) \longrightarrow (Y,\sigma)$ is slightly $\omega b-continuous$ function and Y is clopen Hausdorff, then $G(f)$ is strongly $\omega b-co-closed$ in $X \times Y$.

THEOREM 5.15. Suppose that the function $f:(X,\tau) \longrightarrow (Y,\sigma)$ is an injection and has a strongly $\omega b-co-closed$ graph $G(f)$. Then X is $\omega - T_1$.

PROOF. Let x and y be any two distinct points of X. Then, we have $(x, f(y)) \in [(X \times Y) - G(f)]$. By Lemma 5.12, there exists an $\omega b-clopen$ set U of X and $V \in CO(Y,\sigma)$ such that $(x, f(y)) \in U \times Y$ and $f(U) \cap V = \phi$. Hence $U \cap f^{-1}(V) = \phi$ and $y \notin U$. This implies that X is $\omega b - T_1$.

THEOREM 5.16. Suppose that the function $f:(X,\tau) \longrightarrow (Y,\sigma)$ is a surjection and has a strongly $\omega b-co-closed$ graph $G(f)$. Then Y is $\omega b - T_2$.

PROOF. Let y_1 and y_2 be any distinct points of Y. Since f is surjective. So $f(x) = y_1$ for some $x \in X$ and $(x, y_2) \in (X \times Y) \setminus G(f)$. By Definition 5.11, there exists an $\omega b-clopen$ set U of X and $V \in CO(Y,\sigma)$ such that $(x, y_2) \in U \times V$ and $(U \times V) \cap G(f) = \phi$. Then, we have

$f(U) \cap V = \phi$. Since f is $\omega b-open$, then $f(U)$ is $\omega b-open$ set such that $f(x) = y_1 \in f(U)$. This implies that Y is $\omega b-T_2$.

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REFERENCES

[1] D. Andrijevic, "On b-open sets", Mat. Vesnik., 48 (1996) 59 – 64.
[2] M. Caldas, S. Jafari and T. Noiri, "On Some applications of *b*-open sets in topological spaces", Kochi J. Math. 2(2007), 11 – 19.
[3] C. Carpintero, N. Rajesh, E. Rosas ad S. Saranyasri, "Somewhat $\omega-$continuous functions", Sarajevo Journal of Mathematics, Vol. 11 (23), No.1, (2015), 131 – 137.
[4] Uday Shankar Chakraborty, "On slightly b – continuous functions", Int. J. Contemp. Math. Sciences, Vol. 7 (2012), no. 5, 209 – 213.
[5] E. Ekici and M. Caldas, "Slightly $\gamma-$continuous functions", Bol. Soc. Paran. Mat. , 22(2004), 63 – 74.
[6] K.R. Gentry and H.B. Hoyle, "Somewhat continuous functions", Czechslovak Math. J. 21(96) (1971), 5 – 12. doi: Zbl 0222.54010
[7] H. Z. Hdeib, "$\omega-$ closed mappings", Rev. Colomb. Mat., 16 (1 – 2) (1982), 65 – 78.
[8] H. Z. Hdeib, "$\omega-$ continuous functions", Dirasat Journal 16(2) (1989), 136 – 142.
[9] R. C. Jain, "The role of regularly open sets in general topology", Ph. D. thesis, Meerut University, Institute of Advanced Studies, Meerut, India 1980.
[10] Raja M. Latif, "On Characterizations of Mappings", Soochow Journal of Mathematics, Volume 19, No. 4, pp. 475 – 495. 1993.
[11] Raja M. Latif, "On Semi-Weakly Semi-Continuous Mappings", Punjab University Journal of Mathematics, Volume XXVIII, pp. 22 – 29, (December, 1995).
[12] Raja M. Latif, "Characterizations and Applications of Gamma-Open Sets", Soochow Journal of Mathematics, (Taiwan), Vol. 32, No. 3, pp. 369 – 378. (July, 2006).
[13] Raja M Latif, "Characterizations of Mappings in Gamma-Open Sets", Soochow Journal of Mathematics, (Taiwan), Vol. 33, No. 2, (April 2007), pp. 187 – 202.
[14] Raja M. Latif, "Characterizations of Feebly Totally Open Functions", Advances in Mathematics and Computer Science and their Applications, (2016) pp. 217 – 223.
[15] Raja M. Latif, "Alpha – Weakly Continuous Mappings in Topological Spaces", International Journal of Advanced Information Science and Technology (IJAIST), ISSN # 231 9:2682 Vol. 51, No, 51, July 2016. pp. 12 – 18.
[16] Raja M. Latif, Muhammad Rafiq Raja and Muhammad Razaq, "Contra-Bc-Continuous Functions in Topological Spaces", International Journal of Scientific Research in Science and Technology, Volume 4, Issue 9, (2018), 18 – 29.
[17] Raja M. Latif, "Supra – R – Compactness and Supra – R – Connectedness", Interational Journal of Recent Trends in Engineering & Tchnology, Volum 04, Isse 01, (January – 2018), pp. 426 – 443.
[18] S. N. Maheshwari and S. S. Thakur, "On $\alpha-$ irresolute mappings", Tamkang J. Math. 11 (1980) 209-214.
[19] A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb, "On precontinuous and weak precontinuous mappings", Proc. Math. Phys. soc. Egypt 53 (1982) 47-53.
[20] A. S. Mashhour, I. A. Hasanein and S. N. El-Deeb, "On $\alpha-$ continuous and $\alpha-$ open mappings", Acta Math. Hungarica 41 (1983) 213-218.
[21] O. Njastad, "On some classes of nearly open sets", Pacific J. Math. 15 (1965) 961-970.
[22] T. Noiri and V. Popa, "Almost weakly continuous multifunctions", Demonstratio Math. 26 (1993) 263-280.
[23] T. Noiri and G.I. Chae, "A note on semi – continuous functions", Bull. Calcutta Math Soc. 92(2000), no. 2, 87 – 92.
[24] T. Noiri, "Slightly $\beta-$ continuous functios",IJMMS,28(2001), no. 8, 469 – 478.
[25] Takashi Noiri, Ahmad Al – Omari and Mohd. Salmi Md. Noorani, "On $\omega b-$ open sets and b – Lindeloff spaces", European Journal of Pure and Applied Mathematics, Vol. 1, no. 3, (2008), 3 – 9.
[26] T. Noiri , A. Al – Omani and M.SM. Noorani, "Slightly $\omega-$ continuous functions", Fasciculi Mathematici, No. 41(2009), 97 – 106.
[27] T.M. Nour, "Slightly semi – continuous functions",Bull. Calcutta Math. Soc. 87 (1995), no. 2, 187 – 190.
[28] Manindra Chandra Pal and P. Bhattacharyya, "Faint Precontinuous Functions", Soochow Journal of Mathematics, Volume 21, No. 3, (July 1995), pp. 273 – 289.
[29] A. R. Singal and R. C. Jain, "Slightly continuous mappings", J. Indian Math. Soc. 64 (1997) 195-203.
[30] R. Staum, "The Algebra of bounded continuous functions into a nonachimedean field", Pacific J. Math. 50(1974) 169 – 185.
[31] L. A. Steen and J. A. Seebach Jr, "Counterexamples in Topology", Holt, Rinenhart and Winston, New York 1970.
[32] Albert Wilansky, "Topology for Analysis", Devore Publications, Inc, Mineola New York. (1980).
[33] Stephen Willard, "General Topology, Reading", Mass.: Addison Wesley Pub. Co. (1970).
[34] K. Al – Zoubi and B. Al – Nashef, "The topology of $\omega-$ *subsets*", Al – Manarah. IX (2) (2003), 169 – 179.