Some Topological Separation Axioms Using $g^*b$ - Open Sets

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Abstract - Replacing open sets by $g^*b$ - open sets and ‘cl’ by ‘$g^*b$ - cl’ in $H_i$ - spaces, ($i = 0, 1, 2$) and $U_i$ - Spaces ($i = 0, 1$) of Csaszar [8], we introduce $g^*b - H_i$ - spaces, ($i = 0, 1, 2$) and $g^*b - U_i$ - Spaces ($i = 0, 1$) in topological spaces and study its properties.

Keywords - $g^*b - H_i$ - Spaces ($i = 0, 1, 2$) and $g^*b - U_i$ - Spaces ($i = 0, 1$).

I. INTRODUCTION

Topology has a vital role in pure mathematics and has many subfields. The topology structured the foundation for geometry and algebra. There is no universal agreement among mathematicians as what a first course in topology should include. There are many topics that are appropriate to such a course and not all are equally relevant to the varied purposes.

Separation axioms are properties by which the topology on a space $X$ separates points from points, points from closed sets and closed sets from each other. The various separation axioms give rise to a sequence of successively stronger requirements, which are put upon the topology of a space to separate varying types of subsets.

In 1963, Levine introduced the concept of semi - open sets. Since then, a considerable number of papers discussing separation axioms, essentially by replacing open sets by semi-open sets, have appeared in the literature. For instance, Maheshwari and Prasad introduced semi - $T_0$, semi - $T_1$, semi - $T_2$, s - normality and s - regularity as a generalization of $T_0$, $T_1$, $T_2$, regularity and normality axioms respectively, and investigated their properties. The notion of semi-open sets was used by Maheshwari and Prasad to introduce pairwise semi-$T_0$, pairwise semi - $T_1$, pairwise semi - $T_2$, pairwise s - regular and pairwise s-normal spaces. Moreover, s - normal (resp. semi normal ) spaces were introduced and studied by Maheshwari and Prasad [12] (resp. Dorsett [9]).

Throughout this paper $X$ and $Y$ always represent nonempty topological spaces $(X, \sigma)$ and $(Y, \sigma)$. In this paper, we introduce $g^*b - H_i$ - spaces, ($i = 0, 1, 2$) and $g^*b - U_i$ - Spaces ($i = 0, 1$) in topological spaces and study its properties.

II. $g^*b - H_i$ - SPACES ($i = 0, 1, 2$) AND $g^*b - U_i$ - SPACES ($i = 0, 1$)

The concepts of $g^*b$ - closed sets were introduced and studied by D. Vidhya and R. Parimelazhagan[22] in topological spaces. In this section, we introduce $g^*b - H_i$ - spaces, ($i = 0, 1, 2$) and $g^*b - U_i$ - Spaces ($i = 0, 1$) in topological spaces and study its properties.

Definition 2.1 A subset $A$ of a topological space $(X, \tau)$ is called
1. $g$ - closed [11] if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.
2. $g^*b$ - closed [22], if $bCl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $g$ - open in $X$.

Definition 2.2. A topological space $(X, \tau)$ is said to be:
1. $g^*b - T_0$ [1] if for each pair of distinct points $x, y$ in $X$, there exists a $g^*b$ - open set $U$ such that either $x \in U$ and $y \notin U$ or $x \notin U$ and $y \in U$.
2. $g^*b - T_1$ [1] if for each pair of distinct points $x, y$ in $X$, there exist two $g^*b$ - open sets $U$ and $V$ such that $x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$.
3. \( g^*b - T_2 \) [1] if for each distinct points \( x, y \) in \( X \), there exist two disjoint \( g^*b \) - open sets \( U \) and \( V \) containing \( x \) and \( y \) respectively.

**Definition 2.3**: A space \( X \) is said to be \( g^*b - R_0 \) if for every pair of points \( x \) and \( y \) such that \( x \notin g^*b cl\{y\} \) implies that \( y \notin g^*b cl\{x\} \).

**Definition 2.4**: A space \( X \) is said to be \( g^*b - R_1 \) if for every pair of distinct points \( x, y \) of \( X \) with \( g^*b \) \( cl\{x\} \neq g^*b \) \( cl\{y\} \) there exists a \( g^*b \) - open set \( U \) and a \( g^*b \) - open set \( V \) such that \( x \in U, y \in V, U \cap V = \phi \). Hence, \( X \) is \( g^*b - H_0 \).

**Definition 2.5**: A space \( X \) is said to be \( g^*b - H_0 \) if for every pair of points \( x \) and \( y \) such that \( x \notin g^*b cl\{y\} \) there exists a \( g^*b \) - open set \( U \) and a \( g^*b \) - open set \( V \) such that \( x \in U, y \in V, U \cap V = \phi \).

**Definition 2.6**: A space \( X \) is said to be \( g^*b - H_1 \) if for every pair of points \( x \) and \( y \) such that \( g^*b \) \( cl\{x\} \cap g^*b \) \( cl\{y\} = \phi \), there exists a \( g^*b \) - open set \( U \) and a \( g^*b \) - open set \( V \) such that \( x \in U, y \in V, U \cap V = \phi \).

**Definition 2.7**: A space \( X \) is said to be \( g^*b - H_2 \) if for every \( g^*b \) - closed set \( A \) and a point \( x \) such that \( g^*b \) \( cl\{x\} \cap A = \phi \), there exists a \( g^*b \) - open set \( U \) and a \( g^*b \) - open set \( V \) such that \( x \in U, A \subseteq V, U \cap V = \phi \).

**Definition 2.8**: A space \( X \) is said to be \( g^*b - U_0 \) if for every pair of points \( x \) and \( y \) such that \( x \notin g^*b cl\{y\} \), there exists a \( g^*b \) - open set \( U \) and a \( g^*b \) - open set \( V \) such that \( x \in U, y \in V, g^*b \) \( cl\{U\} \cap g^*b \) \( cl\{V\} = \phi \).

**Definition 2.9**: A space \( X \) is said to be \( g^*b - U_1 \) if for every pair of points \( x \) and \( y \) such that \( g^*b \) \( cl\{x\} \cap g^*b \) \( cl\{y\} = \phi \), there exists a \( g^*b \) - open set \( U \) and a \( g^*b \) - open set \( V \) such that \( x \in V, y \in U, g^*b \) \( cl\{U\} \cap g^*b \) \( cl\{V\} = \phi \).

**Theorem 2.10**: Every \( g^*b \) - normal space is \( g^*b - H_2 \).

**Proof**: Let \( X \) be \( g^*b \) - normal space. Let \( x \in X \) and let \( A \) be a \( g^*b \) - closed set such that \( g^*b \) \( cl\{x\} \cap A = \phi \). By \( g^*b \) - normality of \( X \), there exists a \( g^*b \) - open set \( U \) and a \( g^*b \) - open set \( V \) such that \( g^*b \) \( cl\{x\} \subseteq V, A \subseteq U, U \cap V = \phi \). Hence \( x \in V, A \subseteq U, U \cap V = \phi \). Hence, \( X \) is \( g^*b - H_2 \).

**Theorem 2.11**: Every \( g^*b - H_2 \) space is \( g^*b - H_1 \).

**Proof**: Let \( X \) be \( g^*b - H_2 \) space. Let \( x \) and \( y \) be two distinct points of \( X \) such that \( g^*b \) \( cl\{x\} \cap g^*b \) \( cl\{y\} = \phi \). Since \( X \) is \( g^*b - H_2 \), therefore there exists a \( g^*b \) - open set \( U \) and a \( g^*b \) - open set \( V \) such that \( x \in V, g^*b \) \( cl\{y\} \subseteq U, U \cap V = \phi \). Thus \( x \in V, y \in U, U \cap V = \phi \). Hence, \( X \) is \( g^*b - H_1 \).

**Theorem 2.12**: Every \( g^*b - H_0 \) space is \( g^*b - H_1 \).

**Proof**: Let \( X \) be \( g^*b - H_0 \) space. Let \( x \) and \( y \) be two distinct points of \( X \) such that \( g^*b \) \( cl\{x\} \cap g^*b \) \( cl\{y\} = \phi \). Hence \( x \notin g^*b - cl\{y\} \). Since \( X \) is \( g^*b - H_2 \), therefore there exists a \( g^*b \) - open set \( U \) and a \( g^*b \) - open set \( V \) such that \( x \in V, y \in U, U \cap V = \phi \). Hence, \( X \) is \( g^*b - H_1 \).

**Theorem 2.13**: Every \( g^*b - R_1 \) space is \( g^*b - H_0 \).

**Proof**: Let \( X \) be \( g^*b - R_1 \) space. Let \( x \notin g^*b - cl\{y\} \). Then \( g^*b - cl\{x\} \neq g^*b - cl\{y\} \). Thus there exists a \( g^*b \) - open set \( U \) and a \( g^*b \) - open set \( V \) such that \( x \in U, y \in V, U \cap V = \phi \). Hence, \( X \) is \( g^*b - H_0 \).
Theorem 2.14 : Every $g^*b - R_1$ space is $g^*b - H_1$.
Proof : Follows in view of Theorem 2.12 and 2.13.

Definition 2.15 : A space $X$ is said to be strongly $g^*b$-regular if for each $g^*b$-closed subset $A$ of $X$ and $x \not\in A$, there exist disjoint $g^*b$-open sets $U$ and $V$ such that $x \in U$ and $A \subseteq V$.

Theorem 2.16 : Every $g^*b - H_0$ space is $g^*b - R_0$.
Proof : Let $X$ be a $g^*b - H_0$ space. Let $x \in G \subseteq g^*b - O(\tau)$ and let $y \in X - G$. Then $x \not\in g^*b - cl\{y\}$.
Since $X$ is $g^*b - H_0$, there exists a $g^*b$-open set $U$ and a $g^*b$-open set $V$ such that $x \in U$, $y \in V$, $U \cap V = \emptyset$. Hence $X$ is $g^*b - H_0$.

Theorem 2.17 : Every strongly $g^*b$-regular space is $g^*b - H_2$.
Proof : Let $X$ be a strongly $g^*b$-regular space. Let $x \in X$ and let $A$ be a $g^*b$-closed subset of $X$ such that $g^*b - cl\{x\} \cap A = \emptyset$. Then $x \not\in A$. By strongly $g^*b$-regularity of $X$, there exists a $g^*b$-open set $U$ and a $g^*b$-open set $V$ such that $x \in U$, $A \subseteq V$, $U \cap V = \emptyset$. Hence, $X$ is $g^*b - H_2$.

Theorem 2.18 : A space is $g^*b - T_2$ if and only if it is $g^*b - T_0$ and $g^*b - H_0$.
Proof : Let $X$ be a $g^*b - T_2$ space. Clearly, $X$ is $g^*b - T_0$. Let $x, y \in X$ such that $x \not\in g^*b - cl\{y\}$. Then $x \neq y$, since $X$ is $g^*b - T_2$, there exists a $g^*b$-open set $U$ and a $g^*b$-open set $V$, such that $x \in U$, $y \in V$, $U \cap V = \emptyset$. Hence $X$ is $g^*b - H_0$.
Conversely, let $X$ be a $g^*b - T_0$ and $g^*b - H_0$. Let $x, y$ be two distinct points of $X$. Since $X$ is $g^*b - T_0$, there exists a $g^*b$-open set $U$ or a $g^*b$-open set $V$ such that $x \in U$, $y \not\in V$ or $x \not\in U$, $y \in V$. Thus $x \not\in g^*b - cl\{y\}$ or $y \not\in g^*b - cl\{x\}$. Let $x \not\in g^*b - cl\{y\}$. Since the space is $g^*b - H_0$, there exists a $g^*b$-open set $P$ and a $g^*b$-open set $Q$ such that $x \in P$, $y \in Q$, $P \cap Q = \emptyset$. The result follows similarly in case $y \not\in g^*b - cl\{x\}$. Hence $X$ is $g^*b - T_2$.

Theorem 2.19 : A space is $g^*b - T_2$ if and only if it is $g^*b - T_1$ and $g^*b - H_1$.
Proof : Let $X$ be a $g^*b - T_2$ space. Clearly, $X$ is $g^*b - T_1$. Let $x, y \in X$ such that $g^*b - cl\{x\} \cap g^*b - cl\{y\} = \emptyset$. Then $x, y$ are distinct points of $X$ so that there exists a $g^*b$-open set $U$ and a $g^*b$-open set $V$, such that $x \in U$, $y \in V$, $U \cap V = \emptyset$. Hence, $X$ is $g^*b - H_1$.
Conversely, let $X$ be a $g^*b - T_1$ and $g^*b - H_1$. Let $x, y$ be two distinct points of $X$. Since $X$ is $g^*b - T_1$, therefore $\{x\}$ and $\{y\}$ are $g^*b$-closed sets. Hence $g^*b - cl\{x\} \cap g^*b - cl\{y\} = \emptyset$. Since $X$ is $g^*b - H_1$, there exists a $g^*b$-open set $U$ and a $g^*b$-open set $V$ such that $x \in U$, $y \in V$, $U \cap V = \emptyset$. Hence $X$ is $g^*b - T_2$.

Theorem 2.20 : Every strongly $g^*b$-regular space is $g^*b - U_0$.
Proof : Let $X$ be a strongly $g^*b$-regular space. Let $x, y \in X$ such that $x \not\in g^*b - cl\{y\}$. Since the space is strongly $g^*b$-regular, there exists a $g^*b$-open set $U$ and a $g^*b$-open set $V$ such that $x \in U$, $g^*b - cl\{y\} \subseteq V$, $g^*b - cl\{U\} \cap g^*b - cl\{V\} = \emptyset$. Hence $x \in U$, $y \in V$, $g^*b - cl\{U\} \cap g^*b - cl\{V\} = \emptyset$ and thus the space is $g^*b - U_0$.

Theorem 2.21 : Every $g^*b - U_0$ space is $g^*b - H_0$.
Proof : Let $X$ be a $g^*b - U_0$ space. Let $x, y \in X$ such that $x \not\in g^*b - cl\{y\}$. Since $X$ is $g^*b - U_0$, there exists a $g^*b$-open set $U$ and a $g^*b$-open set $V$ such that $x \in U$, $y \in V$, $g^*b - cl\{U\} \cap g^*b - cl\{V\} = \emptyset$. Hence $x \in U$, $y \in V$, $U \cap V = \emptyset$ and thus $X$ is $g^*b - H_0$. 
Theorem 2.22: Every \( g^* b - U_1 \) space is \( g^* b - H_1 \).

Proof: Let \( X \) is \( g^* b - U_1 \) space. Let \( x, y \in X \) such that \( g^* b - cl \{ x \} \cap g^* b - cl \{ y \} = \emptyset \). Since \( X \) is \( g^* b - U_1 \), there exists a \( g^* b - \) open set \( U \) and a \( g^* b - \) open set \( V \) such that \( x \in U, y \in V, g^* b - cl U \cap g^* b - cl V = \emptyset \). Hence \( x \in V, y \in U, U \cap V = \emptyset \) and thus \( X \) is \( g^* b - H_1 \).

Theorem 2.23: Every \( g^* b \) - normal space is \( g^* b - U_1 \).

Proof: Let \( X \) is \( g^* b \) - normal space. Let \( x, y \in X \) such that \( g^* b - cl \{ x \} \cap g^* b - cl \{ y \} = \emptyset \). Since \( X \) is \( g^* b \) - normal, there exists a \( g^* b - \) open set \( U \) and a \( g^* b - \) open set \( V \) such that \( g^* b - cl \{ x \} \subseteq V, g^* b - cl \{ y \} \subseteq U, g^* b - cl \{ U \} \cap g^* b - cl \{ V \} = \emptyset \). Hence \( x \in V, y \in U, g^* b - cl \{ U \} \cap g^* b - cl \{ V \} = \emptyset \). Hence, \( X \) is \( g^* b - U_1 \).

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